Modal Logic is Dual to Equational Logic

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Abstract

We present a categorical generalisation of the notions of equation and modal formula making these notions dual in the categorical sense. In the case of equations this generalisation is due to Banaschewski and Herrlich [3]. Dualising their results on implicationally definable subcategories (generalised Birkhoff theorems) one obtains co-Birkhoff theorems for modal logics. This should settle—from a semantical (categorical) point of view—the question of co-Birkhoff theorems for coalgebras. Applying these results to Kripke frames yields theorems characterising the expressive power of formulas and rules of infinitary modal logic on Kripke frames. In order to profit from the generality provided by the notion of a coalgebra we also prove characterisation results for the coalgebraic logic of Moss [22].

1 Introduction

This paper contributes to coalgebra (for an overview on recent work see [11, 12]) and infinitary modal logic (for recent work see, e.g., [5, 2, 7, 27]). For work combining coalgebras and modal logics see [5, 22, 24, 19, 14, 13].

Concerning coalgebras, the topics are universal coalgebra (see Rutten [25]) and the search for appropriate logics for specifying coalgebras. Both topics are related by the question of whether an analogue of Birkhoff's variety theorem[1] holds for coalgebras. In order to find an answer to this question one must decide on a logic for coalgebras that could play the role that equational logic has for algebras. Different proposals have been made by Rosu [23], Gumm [10], and Kurz [20]. [23] uses equational logic (for a restricted kind of functors), [10] introduces coequations defined as points of the carrier of a coalgebra, and [20] argues for modal logic. In all these papers, it is only the models (algebras and coalgebras) that are related by categorical duality, not the logics.

Our approach here is essentially driven by semantical considerations: We show that dualising Birkhoff's variety theorem in the form presented by Banaschewski and Herrlich [3] leads to (generalised) modal logic as a dual to equational logic (section 5). It follows that every logic for coalgebras allowing for a categorical dual of Birkhoff's theorem is a (generalised) modal logic in the sense of section 5. This still leaves open the question of Birkhoff-like characterisation results for different kind of logics for coalgebras.

Concerning modal logic, the expressive power of infinitary modal logics on Kripke models was analysed by Barwise and Moss [5, 6] and the expressive power of finitary modal logics on Kripke frames was investigated by Goldblatt and Thomason [8, 9], see also van Benthem [26]. Here, we characterise the expressive power of *infinitary* modal logics on Kripke *frames* (with degree of branching restricted to some infinite regular cardinal) (section 9).

Kripke frames are only a very special case of coalgebras. In order to be able to apply our results to more general types of coalgebras, we prove characterisation theorems also for the coalgebraic logic of Moss [22] (section 8).

Since we want to treat modal logic and coalgebraic logic on a common ground, we distill the essential features in the notion of a modal logic for

[1]Birkhoff's variety theorem states that the equationally definable classes of algebras are precisely the classes of algebras closed under homomorphic images, subalgebras and products (i.e., the varieties).

coalgebras over sets (section 7). Note also that we leave the strict categorical duality in this section: coalgebras over sets are not dual to algebras over sets (but to algebras over complete atomic boolean algebras).

2 Banaschewski and Herrlich [3]

We briefly review the aspects of [3] which are most important for this paper. For details see [3] or [1]. Let $\mathbf{Set}^\Omega$ be the category of algebras for a signature Ω, $F : \mathbf{Set} \to \mathbf{Set}^\Omega$ a free construction and E a set of equations over variables X. Equations E can be categorically characterised by a surjective homomorphism (regular epi) e

$$\begin{tikzcd} FX \arrow[r, "e"] \arrow[dr, "f"'] & FX/E \arrow[d, dotted, "g"] \\ & A \end{tikzcd}$$

where FX is the free algebra over X, FX/E is the quotient of FX w.r.t. to the smallest congruence generated by E, and $e : FX \to FX/E$ is the natural projection. It is not difficult to see that an algebra A satisfies E if and only if for all morphisms $f : FX \to A$ there is a morphism $g : FX/E \to A$ such that the diagram commutes. In categorical terms: A satisfies the equations E iff A is *injective* w.r.t. e.

That is, equations can be generalised to regular epis with a free algebra as domain. The case of implications $\bigwedge_{i \in I}(t_i = s_i \;\Rightarrow\; t = s)$ is even simpler. They just correspond to regular epis. Similarly, implications with finitary premise (Horn formulas) are generalised to epis with finitely presentable domain.

3 Modal Logic and Coalgebras

This section shows how formulas and rules of modal logic can be considered to be certain monomorphism (subcoalgebras) in the corresponding category of coalgebras.

Coalgebras are given w.r.t. a category $\mathcal{X}$ and an endofunctor $\Omega : \mathcal{X} \to \mathcal{X}$. An Ω-coalgebra $A = (UA, \alpha)$ consists of an object $UA \in \mathcal{X}$ and an arrow

$\alpha : UA \to \Omega UA$. Ω-coalgebras form a category $\mathcal{X}_\Omega$ where a morphism $f : (UA, \alpha) \to (UB, \beta)$ is an arrow $f : UA \to UB \in \mathcal{X}$ such that $\Omega f \circ \alpha = \beta \circ f$.

As an example consider $\mathcal{X} = \mathbf{Set}$ and the functor $\Omega X = \mathcal{P}P \times \mathcal{P}_\kappa X$ where P is a set of propositional variables, $\mathcal{P}$ denotes powerset, κ is a cardinal, and $\mathcal{P}_\kappa X = \{Y \subset X : |Y| < \kappa\}$.[2] Then the Ω-coalgebras are the usual *Kripke models* of modal logic: given a coalgebra (UA, α) and a world $x \in UA$, we have $\alpha(x) = (Q, X)$ where $Q \subset P$ and $X \subset UA$. This determines a Kripke model as follows: Q is the set of propositional variables true in the world x and X is the set of worlds accessible from x. The case $P = \{\}$ yields as Ω-coalgebras the *Kripke frames*.

In order to explain the relationship of coalgebras and modal logic, first consider the lefthand diagram below:

$$\begin{tikzcd} FC \arrow[d, "\epsilon_C"'] & \\ C & A \arrow[l, "\gamma"] \arrow[ul, "\gamma^{\#}"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} FC \arrow[d, "\epsilon_C"'] & FC|\varphi \arrow[l, "m_\varphi"'] \\ C & A \arrow[l, "\gamma"] \arrow[ul, "g"'] \arrow[u, "h"'] \end{tikzcd}$$

An Ω-coalgebra FC (and an arrow $\epsilon_C : UFC \to C$) is called *cofree on* $C \in \mathcal{X}$ iff for all Ω-coalgebras A and all arrows $\gamma : UA \to C$ there is a unique morphism $\gamma^{\#} : A \to FC$ such that the diagram commutes.

As an example consider $\Omega = \mathcal{P}_\kappa$, $C = \mathcal{P}P$, P a set of propositional variables. Then A is an Kripke frame with degree of branching smaller than κ. The functions $\gamma : UA \to C$ are valuations: every world in A is assigned the set of propositional variables holding in it. That is, A together with γ, written as (A, γ), is a Kripke model.

Note that, more generally, the $C \times \Omega$-coalgebras can be considered as being pairs (A, γ) where A an Ω-coalgebra and $\gamma : UA \to C$. $C \times \Omega$-morphisms $f : (A, \gamma) \to (B, \delta)$ are Ω-morphisms $f : A \to B$ s.t. $\delta \circ f = \gamma$.

For the second diagram, suppose that a modal formula φ and an Ω-coalgebra (Kripke frame) A are given. According to the definition of satisfaction in modal logic, $A \models \varphi$ iff $A, \gamma, x \models \varphi$ for all $x \in UA$ and all valuations $\gamma : UA \to C$. Let $FC|\varphi$ be the largest $C \times \Omega$-subcoalgebra of (FC, ϵ_C) whose elements satisfy φ and let m_φ be the natural embedding.

[2]This only defines Ω on sets. On functions Ω is defined in the standard way, $\mathcal{P}_\kappa$ being the covariant powerset functor: Given $f : X \to Y$, $\Omega f \equiv \mathrm{id}_{\mathcal{P}P} \times \lambda A \in \mathcal{P}_\kappa X.\{f(a) : a \in A\}$. The cardinality restriction is needed to ensure the existence of cofree coalgebras.

Then the above condition for $A \models \varphi$ can equivalently expressed by saying that for all $\gamma : UA \to C$ there is $h : A \to FC|\varphi$ such that $\epsilon_C \circ m_\varphi \circ h = \gamma$, or equivalently, for all $g : A \to FC$ there is $h : A \to FC|\varphi$ such that $m_\varphi \circ h = g$. That is, A satisfies φ iff A *is projective w.r.t. to* m_φ. Conversely, every subcoalgebra $m : B \to FC$ corresponds to a modal formula if the modal logic under consideration is expressive enough to define the the carrier of B as a subset of the cofree coalgebra.

We have seen that we can generalise modal formulas to embeddings with cofree codomain and modal satisfaction to projectivity. To see how modal rules φ/ψ correspond to just embeddings recall that $A \models \varphi/\psi$ iff

$$\forall \gamma : UA \to C \,.((\forall x \in UA \,.\, A, \gamma, x \models \varphi) \;\Rightarrow\; (\forall x \in UA \,.\, A, \gamma, x \models \psi)).$$

Now consider the following diagram

$$\begin{tikzcd}
FC & FC|\varphi \arrow[l, "m_\varphi"'] & FC|(\varphi \wedge \psi) \arrow[l, "m"'] \\
 & & A \arrow[u, "h"'] \arrow[ul, "g"]
\end{tikzcd}$$

where $FC|\varphi$ and $FC|(\varphi \wedge \psi)$ are the subcoalgebras defined by φ and $\varphi \wedge \psi$ and m_φ, m are the corresponding embeddings. It is not difficult to see that this can equivalently be expressed by saying that for all $g : A \to FC|\varphi$ there is $h : A \to FC|(\varphi \wedge \psi)$ such that $m \circ h = g$. That is, again, A satisfies φ/ψ iff M *is projective w.r.t.* m.

4 M-Coreflective Subcategories

Semantically (i.e., here, categorically), Birkhoff's variety theorem and its relatives can be understood as characterising certain reflective subcategories. And co-Birkhoff-theorems as characterising certain coreflective subcategories. This section reviews some basic facts about this relationship.

Factorisation structures for sinks are an important technical tool to investigate categories of coalgebras because they serve to capture the notions of a subcoalgebra and of a union of images of morphisms. Factorsiation structures for sinks dualise factorisation structures for sources (which serve to describe quotients and intersections of kernels). For more details on factorisation structures see Adamek, Herrlich, Strecker [1].

A **sink** $\mathsf{s} = (A, (s_i)_{i \in I})$ in a category $\mathcal{C}$ consists of an object $A \in \mathcal{C}$ and a family of morphisms $s_i : A_i \to A \in \mathcal{C}$ with common codomain A and indexed by a class I. If I is empty s is called an *empty sink*.

Definition 4.1. Let $\mathcal{E}$ be a collection[3] of sinks and M a class of morphisms in $\mathcal{C}$. $(\mathcal{E}, M)$ is called a **factorisation structure on** $\mathcal{C}$ and $\mathcal{C}$ is called an $(\mathcal{E}, M)$-category iff

1. $\mathcal{E}, M$ are closed under isomorphism.
2. $\mathcal{C}$ has $(\mathcal{E}, M)$-factorisation of sinks, i.e. every sink s in $\mathcal{C}$ has a factorisation $\mathsf{s} = m \circ \mathsf{e}$ for some $m \in M$ and $\mathsf{e} \in \mathcal{E}$.
3. $\mathcal{C}$ has the unique $(\mathcal{E}, M)$-diagonalisation property, i.e. whenever for $m \in M$, $(e_i) \in \mathcal{E}$, $f \in \mathcal{C}$, and a sink (s_i) in $\mathcal{C}$, the square

$$\begin{tikzcd} A \arrow[r, "e_i"] \arrow[d, "s_i"'] & B \arrow[d, "f"] \arrow[dl, dotted, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

commutes for all $i \in I$ then there is a unique diagonal d making the triangles commute.

In a category of coalgebras the notions of subcoalgebra and subobject do not coincide, in general. Our approach is to consider the morphisms in M as subcoalgebras. Moreover, given a sink (s_i) and a factorisation $s_i = m \circ e_i$, the domain of m should be thought of as the union of the images of the s_i (compare the proof of the theorem below).

$\mathcal{B}$ is a M*-coreflective subcategory* of $\mathcal{C}$ iff it is a coreflective subcategory and the coreflection morphisms are in M. The following theorem characterising M-coreflective subcategories can be considered as a first generalised co-Birkhoff theorem. For a complete proof see the proof of its dual in [1], theorem 16.8.

Theorem 4.2. *Let $\mathcal{C}$ be an $(\mathcal{E}, M)$-category and $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. Then $\mathcal{B}$ is M-coreflective iff $\mathcal{B}$ is closed under $\mathcal{E}$-sinks. Moreover, in case*

[3] Since every sink may be indexed by a class there may be more than class-many sinks in $\mathcal{E}$.

that $\mathcal{C}$ has coproducts and is M-wellpowered, this is equivalent to: $\mathcal{B}$ is closed under coproducts and $\mathcal{E}$-quotients.[4]

Proof. For future reference and as a typical example for the use of sinks, we show that closure of $\mathcal{B}$ under $\mathcal{E}$-sinks implies M-coreflectiveness: For $A \in \mathcal{C}$ let (s_i) be the sink consisting of all morphisms with codomain A and domain in $\mathcal{B}$. Let $B_i \stackrel{e_i}{\to} A' \stackrel{m}{\to} A$ be an $(\mathcal{E}, M)$-factorisation of (s_i). Then A' is in $\mathcal{B}$ and m is a M-coreflection morphism for A. □

An important example of an $(\mathcal{E}, M)$-category:

Proposition 4.3. *Let Ω be a functor on* **Set**, *and $A, B \in$* $\mathbf{Set}_\Omega$. *Then $m : A \to B$ is isomorphic to a subcoalgebra of B iff m is a strong mono. Moreover, epi sinks and strong monos form a factorisation structure for sinks on* $\mathbf{Set}_\Omega$.

Definition 4.4 ((enough) M-injective objects). Let $\mathcal{C}$ be a category and M a class of morphisms of $\mathcal{C}$. An object A in a category $\mathcal{C}$ is called M-injective iff for all $m \in M$ and all f there is a g such that $g \circ m = f$. $\mathcal{C}$ is said to have *enough injectives* iff every object is a M-subobject of an injective object.

Proposition 4.5. *The cofree coalgebras in* $\mathbf{Set}_\Omega$ *are StrongMono-injective.*

5 Co-Birkhoff-Theorems

In this section we introduce the notion of a modal logic for an $(\mathcal{E}, M)$-category (see definition 4.1). This notion of a modal formula is categorically dual to the notion of an equation of Banaschewski and Herrlich. Their theorems become theorems characterising the expressive power of (generalised) modal logics.

Definition 5.1 (modal logic for $(\mathcal{E}, M)$-categories).
Let $\mathcal{C}$ be an $(\mathcal{E}, M)$-category. A *rule* is a morphism in M. A *formula* is a morphism in M that has a M-injective codomain. For $A \in \mathcal{C}$, $m : B \to B' \in$

[4] *M-wellpowered* means that—up to isomorphism—every $M \in \mathcal{C}$ has only a *set* of M-subobjects (M-morphisms with codomain M). *Closure under $\mathcal{E}$-quotients* means: for all $M \in \mathcal{B}$ and every sink $e : M \to M'$ in $\mathcal{E}$, also $M' \in \mathcal{B}$.

M and a morphism $f : A \to B'$ we define: $A, f \models m$ iff there is $g : A \to B$ such that $m \circ g = f$:

$$\begin{tikzcd} B' & B \arrow[l, "m"'] \\ & A \arrow[ul, "f"] \arrow[u, dotted, "g"'] \end{tikzcd}$$

We define $A \models m$ iff $A, f \models m$ for all f, that is, iff A is projective w.r.t. m. For $\Phi \subset M$, $\mathsf{Mod}(\Phi)$ denotes the full subcategory of $\mathcal{C}$ induced by the objects satisfying every rule in Φ.

Definition 5.2 (definable subcategories). We call a full subcategory $\mathcal{B}$ of $\mathcal{C}$ *rule definable* iff $\mathcal{B} = \mathsf{Mod}(\Phi)$ for some $\Phi \subset M$. We call $\mathcal{B}$ *modally definable* iff $\mathcal{B} = \mathsf{Mod}(\Phi)$ for some $\Phi \subset M$ consisting only of formulas.

A first example of a rule definable class are the models of all rules: the empty sinks in $\mathcal{E}$.

Example 5.3. Let $\mathcal{E}_{\{\}} = \{A \in \mathcal{C} : (A, \{\}) \in \mathcal{E}\}$ be the class of empty sinks in $\mathcal{E}$. Then $\mathcal{E}_{\{\}} = \mathsf{Mod}(M)$.

Note that, by applying (2) of definition 4.1 to an empty sink, it follows that $\mathsf{Mod}(M)$ is never empty (assuming that $\mathcal{C}$ is not empty). This seems to be counterintuitive: The class of all modal formulas is generally inconsistent and has an empty class of models. The reason for this mismatch is that, modally speaking, we did not exclude the empty Kripke frame as a possible model.

We can now dualise the results of Banaschewski and Herrlich [3]. We sketch the proofs for future reference. First, the dual of [3], proposition 1 (also [1], theorem 16.14).

Theorem 5.4 (rule definable subcategories). *Let $\mathcal{C}$ be an $(\mathcal{E}, M)$-category. Then a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is rule definable iff $\mathcal{B}$ is closed under $\mathcal{E}$-sinks.*

Proof. It is easy to see that rule definable implies closure under $\mathcal{E}$-sinks. For the converse: By theorem 4.2, closure under $\mathcal{E}$-sinks implies M-coreflectiveness. To see that this implies definability let $\Phi = \{m \mid m \text{ coreflection morphism}\}$ and show $\mathcal{B} = \mathsf{Mod}(\Phi)$. $\square$

And the dual of [3], proposition 3 (also [1], theorem 16.17).

Theorem 5.5 (modally definable subcategories). *Let $\mathcal{C}$ be an $(\mathcal{E}, M)$-category with enough injectives. Then a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is modally definable iff $\mathcal{B}$ is closed under $\mathcal{E}$-sinks and M-subobjects.*

Proof. Let $\Phi = \{m : A \to B \mid m$ coreflection morphism and B M-injective$\}$. Rest as above. $\square$

Why do we not state a theorem dualising proposition 2 of [3]? Proposition 2 in [3] characterises classes definable by implications with fewer than κ premises (for some infinite regular cardinal κ). These are those classes closed under monosources and κ-filtered colimits. The proof uses the reflection morphisms with κ-presentable domain.

The dual of this proposition does not apply to categories of coalgebras over set. The reason is that the notions dual to κ-filtered colimit and κ-presentable object do not have the necessary properties in **Set**.

6 On Formulas and Rules as Morphisms

In this section, we have a look at the interplay of logical notions concerning formulas and categorical notions concerning morphisms. We fix an $(\mathcal{E}, M)$-category $\mathcal{C}$.

Let $e : A_2 \to A_1 \in \mathcal{C}$ and $m_2, m_1 \in M$ with codomain A_2, A_1, respectively. Then m_2, m_1 are called **equivalent w.r.t. e** iff for all $B \in \mathcal{C}$, $f : B \to A_2$ it holds $B, f \models m_2 \;\Leftrightarrow\; B, e \circ f \models m_1$. m_2, m_1 are called **equivalent** iff for all $B \in \mathcal{C}$, $B \models m_2 \;\Leftrightarrow\; B \models m_1$. A morphism e in $\mathcal{C}$ is called **true** iff all objects of $\mathcal{C}$ are projective w.r.t. e. A pullback of a true morphism is true and $m \in M$ is true iff m is iso. Equivalence w.r.t. a true morphism implies equivalence (but not vice versa). A morphism $m : A \to B \in M$ is called **invariant** iff $A \models m$. In the case of $\mathcal{C} = \mathbf{Set}$ and m an inclusion this means that for all $f : A \to B$, $f(A) \subset A$.

An important fact on equivalence of morphisms is the following:

Proposition 6.1. *Let $m_2, m_1 \in M$, $e \in \mathcal{C}$ and consider the following diagram:*

$$\begin{tikzcd}
\bullet \arrow[d, "e"'] & \bullet \arrow[l, "m_2"'] \arrow[d, dotted, "e'"] \\
\bullet & \bullet \arrow[l, "m_1"]
\end{tikzcd}$$

There is (a unique) e' making the diagram into a pullback iff m_2, m_1 are equivalent w.r.t. e. Moreover, if e is true and the diagram a pullback then m_2, m_1 are equivalent.

For $m_1, m_2 \in M$ with common codomain B, we define

$$A \models m_1/m_2 \qquad \text{iff} \qquad \forall f : A \to B\,.\ A, f \models m_1 \ \Rightarrow\ A, f \models m_2.$$

If $m_1, m_2 \in M$ are formulas then m_1/m_2 is called a **rule**.

Proposition 6.2. *Let $m, m_1, m_2 \in M$ with $m_2 = m_1 \circ m$ and B the codomain of m_1. Then $A \models m$ iff $A \models m_1/m_2$.*

Corollary 6.3. *Let $\mathcal{C}$ have enough injectives. Then for all $m \in M$ there are formulas $m_1, m_2 \in M$ such that $A \models m$ iff $A \models m_1/m_2$.*

Conjunction is pullback: Let $m_1, m_2 \in M$ and

$$\begin{tikzcd}
\bullet \arrow[d, "m_1"'] & \bullet \arrow[l, "m_2'"'] \arrow[d, "m_1'"] \\
B & \bullet \arrow[l, "m_2"]
\end{tikzcd}$$

be a pullback. Define $m_1 \wedge m_2 = m_1 \circ m_2'$.

Proposition 6.4. *Suppose that the diagram above is a pullback. Then for all $f : A \to B$, $A, f \models m_1 \wedge m_2$ iff $A, f \models m_1$ and $A, f \models m_2$. Moreover, $A \models m_1/m_2$ iff $A \models m_1/(m_1 \wedge m_2)$ iff $A \models m_2'$.*

It follows:

Corollary 6.5. *Let $\mathcal{C}$ have pullbacks. Then for all formulas $m_1, m_2 \in M$ there is $m \in M$ such that $A \models m_1/m_2$ iff $A \models m$.*

It follows from the two corollaries that for categories $\mathcal{C}$ with pullbacks and enough injectives the notion of a rule as a morphism in M and of a rule as m_1/m_2 for formulas $m_1, m_2 \in M$ are equivalent.

7 Modal Logics for Coalgebras over Set

This section investigates modal logics for categories $\mathbf{Set}_\Omega$ whose forgetful functor $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ has a right adjoint F. As injectives we can now take the coalgebras FC cofree over C. A formula with colours (variables) in C will be interpreted as a subcoalgebra of FC. Moreover, given a coalgebra A, there is now a one-to-one correspondence between colourings (valuations of variables) $UA \to C$ and morphisms $A \to FC$.

Recall propositions 4.3, 4.5. Also note that the remark after corollary 6.5 applies since $\mathbf{Set}_\Omega$ is complete (see [17]).

Given a signature Ω, we denote by M the class of strong monos (i.e. subcoalgebras) of $\mathbf{Set}_\Omega$ and by M_C the class of subcoalgebras of the cofree coalgebra FC. We define a **modal logic for Ω-coalgebras** $\mathcal{L}$ to consist of the following:

1. a category *Col* (of sets of colours) which is a thin[5] subcategory of **Set** with the arrows $p : D \to C \in \mathit{Col}$ being surjections,

2. a class of formulas $\mathcal{L}_C$ for each $C \in \mathit{Col}$. $\mathcal{L}_C \subset \mathcal{L}_D$ iff there is $p : D \to C \in \mathit{Col}$, $\mathcal{L}_C \cap \mathcal{L}_D = \{\}$ otherwise,

3. a function $\mathit{Morph}_C : \mathcal{L}_C \to M_C$ for each $C \in \mathit{Col}$,

and, for each $p : D \to C$ in *Col* and $\varphi \in \mathcal{L}_C$, we require that

4. $\mathit{Morph}_D(\varphi)$ is a pullback of $\mathit{Morph}_C(\varphi)$ along Fp.

Furthermore, define for a coalgebra A, a valuation $\gamma : UA \to C$ and a formula $\varphi \in \mathcal{L}_C$: $A, \gamma \models \varphi \;\Leftrightarrow\; A, \gamma^\# \models \mathit{Morph}_C(\varphi)$ (see definition 5.1). $A \models \varphi$ is defined by quantifying over all $\gamma : UA \to C$. For $\varphi, \psi \in \mathcal{L}_C$, φ/ψ is called a rule of $\mathcal{L}$. We define $\mathit{Morph}_C(\varphi/\psi)$ to be the pullback of $\mathit{Morph}_C(\varphi)$ along $\mathit{Morph}_C(\psi)$ (compare with the diagram before proposition 6.4).

To explain the definition recall section 3. Let Ω be $\mathcal{P}_\kappa$ and $\mathbf{P}$ a class of propositional variables. The objects of *Col* are the sets $\mathcal{P}P$ for $P \subset \mathbf{P}$. The morphisms are $\iota^{-1} : \mathcal{P}Q \to \mathcal{P}P$ where $\iota : P \hookrightarrow Q$. $\mathcal{L}_{\mathcal{P}P}$ is the set of formulas of infinitary modal logic with variables from P. For a formula $\varphi \in \mathcal{L}_{\mathcal{P}P}$, $\mathit{Morph}(\varphi)$ is the largest $\mathcal{P}P \times \Omega$-subcoalgebra of $(F\mathcal{P}P, \epsilon_{\mathcal{P}P})$ whose carrier is contained in $\{x \in UF\mathcal{P}P : F\mathcal{P}P, \epsilon_{\mathcal{P}P}, x \models \varphi\}$. As a consequence, for a Kripke

[5]There is at most one arrow between any two objects.

frame A, a valuation $\gamma : UA \to \mathcal{P}P$ and $\varphi \in \mathcal{L}_{\mathcal{P}P}$, the definition of $A, \gamma \models \varphi$ given above coincides with the definition of $\models$ as given by standard modal logic. Condition (4) holds because standard modal logic has the following property: given a formula φ with variables in P, given $\iota : P \hookrightarrow Q$ and a Kripke frame A, the relation $A \models \varphi$ does not depend on whether one interprets φ as a formula of $\mathcal{L}_{\mathcal{P}P}$ or $\mathcal{L}_{\mathcal{P}Q}$. For a proof of condition (4) use proposition 6.1.

Note that, according to (3), for $\varphi \in \mathcal{L}_C$ and $\mathcal{L}_C \subset \mathcal{L}_D$, $\mathit{Morph}_C(\varphi)$ and $\mathit{Morph}_D(\varphi)$ may be different morphisms. But due to (4) and proposition 6.1, $\mathit{Morph}_C(\varphi)$ and $\mathit{Morph}_D(\varphi)$ are equivalent (since Fp is true when p surjective).

An important consequence of our definition of a modal logic for Ω-coalgebras is that formulas are invariant under $C \times \Omega$-bisimulations. This is almost immediate from $\mathbf{Set}_\Omega$ being an $(\mathit{Epi}, \mathit{StrongMono})$-category:

Proposition 7.1. *Let $f : A \to B$ be epi in $\mathbf{Set}_\Omega$, $C \in \mathit{Col}$ and $\gamma : UA \to C$, $\delta : UB \to C$ with $\delta \circ f = \gamma$. Then for all $\varphi \in \mathcal{L}_C$: $A, \gamma \models \varphi \;\Leftrightarrow\; B, \delta \models \varphi$.*

A modal logic for Ω-coalgebras $\mathcal{L}$ is called **expressive** iff for all $C \in \mathbf{Set}$ and all invariant $m \in M_C$ there is $\varphi \in \mathcal{L}$ such that $\mathit{Morph}(\varphi)$ and m are equivalent. This condition does not guarantee that all morphisms in M can be expressed as rules of $\mathcal{L}$. Therefore, a modal logic for Ω-coalgebras $\mathcal{L}$ is called **rule-expressive** iff for all $C \in \mathbf{Set}$ there is a surjection $p : D \to C$ such that for all $m \in M_C$ there is $\varphi \in \mathcal{L}_D$ such that $\mathit{Morph}(\varphi)$ and m are equivalent w.r.t. Fp. It follows[6] that for every $m \in M$ there are $\varphi, \psi \in \mathcal{L}$ such that m and $\mathit{Morph}(\varphi/\psi)$ are equivalent.

We now obtain as a corollary to theorems 5.4 and 4.2:

Theorem 7.2 (co-quasivariety theorem). *Let Ω be an endofunctor on $\mathbf{Set}$ such that the forgetful functor $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ has a right adjoint and let $\mathcal{L}$ be a rule-expressive modal logic for Ω-coalgebras. Then a full subcategory $\mathcal{B}$ of $\mathbf{Set}_\Omega$ is definable by rules of $\mathcal{L}$ iff $\mathcal{B}$ is closed under quotients and coproducts.*

Proof. We sketch "if": Recall the definition of Φ in the proof of 5.4. We have to show that for every $m \in \Phi$ there are $\varphi, \psi \in \mathcal{L}$ such that m is equivalent to $\mathit{Morph}(\varphi/\psi)$. But this follows from expressiveness of $\mathcal{L}$. □

[6] Using corollary 6.3 and propositions 6.1, 6.2 and 6.4.

Similarly, as a corollary to theorems 5.5 and 4.2 (note that the morphisms in the class Φ appearing in the proof of theorem 5.5 are invariant):

Theorem 7.3 (covariety theorem). *Let Ω be an endofunctor on* **Set** *such that the forgetful functor $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ has a right adjoint and let $\mathcal{L}$ be an expressive modal logic for Ω-coalgebras. Then a full subcategory $\mathcal{B}$ of $\mathbf{Set}_\Omega$ is definable by formulas of $\mathcal{L}$ iff $\mathcal{B}$ is closed under subcoalgebras, quotients and coproducts.*

In the case that the functor Ω is bounded[7] by a set $C \in Col$ we can strengthen the last result in that we need only formulas from $\mathcal{L}_C$ to achieve expressiveness. This is due to the following proposition.

Proposition 7.4. *Let Ω be bounded by $C \in Col$ and let $\mathcal{L}$ be a modal logic for Ω-coalgebras such that for every $m_C \in M_C$ there is $\varphi \in \mathcal{L}_C$ such that m_C and $Morph(\varphi)$ are equivalent. Then $\mathcal{L}$ is expressive.*

Proof. (Hint) Let $m_D \in M_D$. We have to show that there is an equivalent $m_C \in M_C$. For $|D| < |C|$ let m_C be the pullback of m_D along Fp for a surjection $p : D \to C$ and use proposition 6.1. For $|D| > |C|$ (more complicated) use that every coalgebra is the union of coalgebras of cardinality $\leq |C|$ each of which can be embedded in FC. $\square$

8 Coalgebraic Logic

The coalgebraic logic of Moss [22] gives us a logic for coalgebras for *any* standard and weak-pullback preserving functor Ω on **Set**. It is the purpose of this section to show that coalgebraic logic is a (rule-expressive) modal logic for coalgebras in the sense of the previous section.

We briefly recall the definition of coalgebraic logic of Moss [22].

Definition 8.1 (coalgebraic logic, syntax). The language $\mathcal{CL}_\Omega$ is defined to be the least class satisfying:

$$\begin{array}{lcl} \Phi \subset \mathcal{CL}_\Omega,\ \Phi \text{ a set} & \Longrightarrow & \bigwedge \Phi \in \mathcal{CL}_\Omega \\ \varphi \in \mathcal{CL}_\Omega & \Longrightarrow & \neg\varphi \in \mathcal{CL}_\Omega \\ \varphi \in \Omega(\mathcal{CL}_\Omega) & \Longrightarrow & \varphi \in \mathcal{CL}_\Omega \end{array}$$

[7] A functor Ω is *bounded by* C iff for every Ω-coalgebra A and every $a \in UA$ there is a subcoalgebra B of A such that $a \in UB$ and the cardinality of UB is not greater than the cardinality of C, see [16, 25].

The last clause uses the fact that Ω can be extended uniquely to a set-based functor on the category **SET** of classes and set-continious functions.

Definition 8.2 (coalgebraic logic, semantics). Given a coalgebra A the semantics is given by the least relation $\models_\Omega \subset UA \times \mathcal{CL}_\Omega$ such that (let $x \in UA$):

$$\begin{array}{lll} x \models_\Omega \varphi \text{ for all } \varphi \in \Phi, \Phi \subset \mathcal{CL}_\Omega,\ \Phi \text{ a set} & \Rightarrow & x \models_\Omega \bigwedge \Phi \\ x \not\models_\Omega \varphi & \Rightarrow & x \models_\Omega \neg\varphi \\ \text{there is } w \in \Omega(\models_\Omega) \text{ s.t. } \Omega\pi_1(w) = f(x), \Omega\pi_2(w) = \varphi & \Rightarrow & x \models_\Omega \varphi \end{array}$$

where π_1, π_2 denote the projections from the product $UA \times \mathcal{CL}_\Omega$ to its components. The last clause makes use of $\models_\Omega$ being in **SET**.

The next step is to add variables to the logic. Another possibility to do this (and which is sufficient for a co-variety theorem) was given in [20].

Definition 8.3 (coalgebraic logic with propositional variables). Let $\mathbf{P}$ a class of propositional variables, and P be a sub*set* of $\mathbf{P}$. The coalgebraic logic $\mathcal{CL}_{P,\Omega}$ for Ω with variables from P, is given as in definition 8.1 plus a clause for propositional variables:

$$p \in P \quad \Longrightarrow \quad p \in \mathcal{CL}_{P,\Omega}.$$

$\mathcal{CL}_{\mathbf{P},\Omega}$ is $\bigcup\{\mathcal{CL}_{P,\Omega} : P \subset \mathbf{P}, P \text{ a set}\}$. A model is given by a $\mathcal{P}P \times \Omega$-coalgebra, i.e., an Ω-coalgebra A and a valuation $\gamma : UA \to \mathcal{P}P$. The semantics $\models_{P,\Omega}$ is given as in definition 8.2 plus

$$p \in \gamma(x) \quad \Longrightarrow \quad x \models_{P,\Omega} p.$$

For $\varphi \in \mathcal{CL}_{\mathbf{P},\Omega}$ and $x \in UA$, we define $A, x \models_{\mathbf{P},\Omega} \varphi$ iff for some $P \subset \mathbf{P}$ such that $\varphi \in \mathcal{CL}_{P,\Omega}$ it holds $\forall \gamma : UA \to \mathcal{P}P\,.\, A, \gamma, x \models_{P,\Omega} \varphi$.

It is not difficult to see that $\mathcal{CL}_{\mathbf{P},\Omega}$ meets the requirements of a modal logic for coalgebras as in section 7. To ensure that it is (rule-)expressive we require the functors $\mathcal{P}P \times \Omega$ to be uniform. In [22], *uniform* functors are defined and it is proven that for uniform functors Ω the logic $\mathcal{CL}_\Omega$ has characterising formulas. This can be used to deduce from theorem 7.3:

Theorem 8.4 (co-quasivariety theorem). *Let Ω be a standard and weak-pullback preserving functor on* **Set** *such that $\mathcal{P}P \times \Omega$ is uniform for all sets P and such that $U : \mathbf{Set}_\Omega \to$* **Set** *has a right adjoint. Then a class K of Ω-coalgebras is definable by rules of $\mathcal{CL}_{\mathbf{P},\Omega}$ iff it is closed under quotients and coproducts.*

For a covariety theorem for coalgebraic logic see [20].

9 Modal Logic for Kripke Frames

In this section we apply the results of section 7 to analyse the expressive power of infinitary modal logics on Kripke frames (with degree of branching restricted to some infinite regular cardinal).

For the remainder of this section let κ be an infinite regular cardinal. We denote with $\mathbf{KF}_\kappa$ the class of Kripke frames with degree of branching smaller than κ, i.e., $\mathbf{KF}_\kappa$ is $\mathbf{Set}_{\mathcal{P}_\kappa}$. Note that $\mathcal{P}_\kappa$ is bounded[8] by κ (for $\kappa = \aleph_{\alpha+1}$ for some ordinal α, $\mathcal{P}_\kappa$ is even bounded by $\aleph_\alpha$). The proof of these facts is similar to an argument in the proof of proposition 1.3 in Barr [4].

We denote with $\mathcal{ML}_\infty$ the infinitary modal logic built from a proper class of propositional variables, the constant $\bot$, the operators $\neg, \Box$ and conjunctions $\bigwedge$ over any set of formulas. $\bigvee$ and $\Diamond$ are defined as the usual abbreviations. $\mathcal{ML}_\kappa(P)$ denotes the subset of $\mathcal{ML}_\infty$ where conjunctions are taken only over sets of cardinality not greater than κ and where propositional variables are from the set P.

The following fact on infinitary modal logic follows from Baltag's lemma in Barwise and Moss [5], 11.13.

Lemma 9.1. *Let κ be an infinite regular cardinal, $|P| \leq \kappa$, $A \in \mathbf{KF}_\kappa$, $\alpha : UA \to \mathcal{P}P$, $a \in A$. Then there is $\varphi^a \in \mathcal{ML}_\kappa(P)$ such that for all $B \in \mathbf{KF}$, $\beta : UB \to \mathcal{P}P$, $b \in B$ it holds that $B, \beta, b \models \varphi^a$ only if (A, α, a) and (B, β, b) are bisimilar.*

We can now show the following theorem which has appeared in a weaker form and with a different proof in [20].

Theorem 9.2 (definability by rules). *Let κ be an infinite regular cardinal and κ' be the cardinal of the carrier of the final $\mathcal{P}_\kappa$-coalgebra. Then $\mathcal{B} \subset \mathbf{KF}_\kappa$ is definable by rules of $\mathcal{ML}_\infty$ iff $\mathcal{B}$ is closed under subframes and disjoint unions.*

Proof. $\mathcal{ML}_\infty$ is a modal logic for $\mathcal{P}_\kappa$-coalgebras in the sense of section 7. As explained there, the objects of *Col* are $\{\mathcal{P}P : P \text{ a set}\}$. The morphisms are $\iota^{-1} : \mathcal{P}Q \to \mathcal{P}P$ where $\iota : P \hookrightarrow Q$. $\mathcal{L}_{\mathcal{P}P}$ is $\mathcal{ML}_\infty(P)$. $\mathcal{ML}_\infty$ is the union of all $\mathcal{ML}_\infty(P)$, $P \in Col$. That $\mathcal{ML}_\infty$ is rule-expressive follows from the lemma. Now apply theorem 7.2. $\square$

[8] See footnote 7 for a definition.

As for algebras, in the case of definability by formulas (equations) we can restrict the cardinality of the set of colours (variables) and the cardinality of the disjunctions (number of equations). (A less categorical proof of the following theorem in the style of [20] is available as [18].)

Theorem 9.3 (definability by formulas). *Let κ be an infinite regular cardinal, P a set with $|P| \leq \kappa$ and $|\mathcal{P}P| \geq \kappa$ and κ' be the least regular cardinal which is larger than the carrier of the cofree $\mathcal{P}_\kappa$-coalgebra over $\mathcal{P}P$. Then $\mathcal{B} \subset \mathbf{KF}_\kappa$ is definable by formulas of $\mathcal{ML}_{\kappa'}(P)$ iff $\mathcal{B}$ is closed under subframes, disjoint unions and p-morphic images*[9].

Proof. To apply theorem 7.3 we need to show that $\mathcal{ML}_{\kappa'}(P)$ is expressive. According to proposition 7.4 we have to show that every subcoalgebra A of the cofree coalgebra $F\mathcal{P}P$ can be expressed by a $\varphi \in \mathcal{ML}_{\kappa'}(P)$. For all $a \in A$ let φ^a be as in the lemma. Let $\varphi = \bigvee_{a \in A} \varphi^a$. Since $\kappa' \geq \kappa$, $\varphi \in \mathcal{ML}_{\kappa'}(P)$. $\square$

Obviously, this theorem implies that a class of frames in $\mathbf{KF}_\kappa$ is $\mathcal{ML}_\infty$-definable iff it is closed under subframes, disjoint unions and p-morphic images. But it also shows why infinitary logic is needed: First, we need enough propositional variables P such that every generated subframe can be embedded into $F\mathcal{P}P$ (see the proof of proposition 7.4). Second, we need infinitary disjunctions of the cardinality of $F\mathcal{P}P$ to achieve expressiveness.

It seems that in order to know more about the size of $F\mathcal{P}P$ we have to make further set-theoretic assumptions. For example, it follows from Martin's axiom (or the stronger continuum hypothesis) that 2^ω is regular (see Martin and Solovay [21]). A calculation similar to the one of the proof of proposition 1.3 in Barr [4] shows that, in the case of $\kappa = \omega$, we can then choose $P = \omega$ and $\kappa' = 2^\omega$. Similarly, assuming the general continuum hypothesis, in the case of $\kappa = \aleph_{\alpha+1}$ for some ordinal α, we can choose $P = \aleph_\alpha$ and $\kappa' = \aleph_{\alpha+1}$.

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[9] *p-morphism* is the traditional name in the literature on modal logic for morphisms in $\mathbf{Set}_\mathcal{P}$.

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