Dualities from Double Categories of Relations

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The purpose of this note is to show how various duality theorems for so-called subordinations can be obtained from a "master theorem" that extends Priestley duality to double categories of relations. We write $\mathsf{DL}, \mathsf{BA}, \mathsf{Pri}, \mathsf{Stone}$ for the categories of bounded distributive lattices, Boolean algebras, Priestley and Stone spaces, respecticely. The notion of a **$\mathbb{B}$-relation** for a concrete order-enriched category $\mathbb{B}$ (satisfying mild conditions) induces a **double category of $\mathbb{B}$-relations** $\mathbb{R}$ as follows, see [6] and the slides for the presentation at BLAST 2022.

- $\mathbb{R}$ has as objects $\mathbb{B}$-relations $R : A \looparrowright B$ with $A, B \in \mathbb{B}$. A **$\mathbb{B}$-relation** R can be tabulated in $\mathbb{B}$, that is, represented as a 'span of projections' $(\mathbf{R} \to A, \mathbf{R} \to B)$. The span must be jointly mono. Moreover, relations are **weakening**, that is $x' \le xRy \le y' \;\Rightarrow\; x'Ry'$.

- A pair of maps $(f : X \to X', g : Y \to Y')$ in $\mathbb{B}$ is an arrow $R \to R'$ in $\mathbb{R}$ if $xRy \;\Rightarrow\; f(x)R'g(y)$. An arrow $(f, g) : R \to R'$ is also called a **square** and we draw f, g vertically and R, R' horizontally. Relations are ordered since $(id, id) : R \to R'$ is a square iff $R \subseteq R'$.

- **Vertical composition** in $\mathbb{R}$ is inherited from $\mathbb{B}$. **Horizontal composition** is composition of relations.

Master Theorem [6]: The dual equivalence between Pri and DL extends to Pri-relations and DL-relations. The dual of a DL-relation $\prec : A \looparrowright B$ is $R : X \looparrowright Y$ where X, Y are the Priestley spaces of prime filters of A, B, respectively, and

$$xRy \;\Leftrightarrow\; \prec[x] \subseteq y,$$

that is, xRy iff $a \in x$ and $a \prec b$ implies $b \in y$. The dual of a Pristley-relation $R : X \looparrowright Y$ is $\prec : A \looparrowright B$, where A, B are the DLs of upper clopens of X, Y, respectively, and

$$a \prec b \;\Leftrightarrow\; R[a] \subseteq b,$$

that is, $a \prec b$ iff $x \in a$ and xRy implies $y \in b$.

Remark: This duality does not flip relations but the order between relations. From the double theoretic point view relations are better understood as objects in $\mathbb{R}$ rather than arrows.

We are interested in restricting this duality to endo-relations, that is, to categories where, on the one hand, objects are of the kind (X, R) with X a Priestley space and R a relation on X and where, on the other hand, objects are of the kind $(B, \prec)$ with B a distributive lattice and $\prec$ is a relation on B. The latter ones are known as proximities or subordinations in the literature, see eg [1].

A **subordination** $\prec$ on a distributive lattice (or a Boolean algebra) B is a DL-relation $B \looparrowright B$. The (order-enriched) category SubDL of **subordination algebras** is the (vertical) subcategory of the double category of DL-relations with subordination algebras $(B, \prec)$ as objects and arrows $f : B \to B'$ with $a \prec b \;\Rightarrow f(a) \prec' f(b)$. The category of **BA-based subordinations** SubBA further restricts to Boolean algebras B.

We now go through a list of corollaries to the master theorem.

Corollary: The category of Priestley spaces with closed weakening relations is dually equivalent to SubDL.

The next corollary is a variation of [3, Thm.9] and [5, Thm.3], see [1, Remark 2.23].

Corollary: The category SubBA is dually equivalent to the category of Stone spaces with closed relations.

The next examples exploit that duality of relations reverses the order between relations. For exmple, *reflexive* is dual to *below idenity*, *transitive* is dual to *interpolative*.

Corollary: The category of Stone spaces with reflexive closed relations is dually equivalent to the category of BA-based subordinations below identity.

Corollary: The category of Stone spaces with transitive closed relations is dually equivalent to the category of BA-based interpolative subordinations.

Corollary: The category of Stone spaces with reflexive and transitive closed relations (aka preorders, quasi-orders) is dually equivalent to the category of BA-based interpolative subordinations below identity.

A **descriptive general frame** (X, R) is a Stone space with a closed relation R such that R^{-1} preserves clopens. The inverse image of R preserves clopens iff, for all clopens b, the downsets $\{a \mid a \prec b\}$ of the dual relation $\prec$ have a maximal element (given by $\Box_R b$). Such a relation $\prec$ is called **modally definable**, see [1, Def.2.9,Lem.4.3]. We now obtain [1, Thm.4.5(1)] as another corollary.

Corollary: The category of BA-based modally definable subordination algebras is dually equivalent to the category of DGF's with relational morphisms.

The constraints stemming from the order of relations and the constraint of preservation of clopens/modal definability can be combined in a modular fashion:

Corollary: The category of reflexive DGF's with relational morphisms is dually equivalent to the category of modally definable BA-based subordinations below identity.

Corollary: The category of transitive DGF's with relational morphisms is dually equivalent to the category of modally definable BA-based interpolative subordinations.

Corollary: The category of reflexive and transitive DGF's with relational morphisms is dually equivalent to the category of modally definable BA-based interpolative subordinations below identity.

We phrased the results in terms of the more widely considered BA-based subordinations but all of our proofs are obtained by restricting more general results about (DL-based) subordination algebras.

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References

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