Stone Duality for Relations

Alexander Kurz, Drew Moshier, Achim Jung

September 22, 2020

Dual Relations

Given a topological space extended with

- an equivalence relation or preorder, what is the algebraic structure dual to the quotient of the space?
- a non-determinsic computation (relation), what is the dual relation between pre- and post-conditions?

Given an algebraic structure extended with

- relations, what is the topological dual?

Given an (in)equational calculus of logical operations extended with

- a Gentzen $\vdash$, what is its dual semantics for which it is sound and complete?

Motivating Example: Cantor Space

Cantor Space C : Middle Third Subset of the Unit Interval

Equivalence $\equiv$ relation glueing together the endpoints of the gaps

Dual of Cantor Space: Free Boolean algebra A over the set $\mathbb{N}$

What is the dual $\prec$ of $\equiv$?

What is the dual of $(A, \prec)$?

This talk concentrates on the first question

The second question is the subject of another talk. In brief:

- The dual of $(A, \prec)$ is the unit interval
- Compact ordered Hausdorff spaces arise from splitting idempotents in the category of Priestley spaces with relations as arrows, with the duality mediated by so-called round filters

Example: Priestley Spaces

Stone space C given

Relation $\leq$ such that $(C, \leq)$ a Priestley space given

Then:

Two clopens a, b are in the dual of $\leq$ if $\uparrow a \subseteq b$

The reflexive elements $\uparrow a \subseteq a$ are the upper clopens

$(C, \leq)$ is the coinserter of C wrt to $\leq$

The dual of $(C, \leq)$ is the distributive lattice of reflexive elements

The dual of $(C, \leq)$ is the inserter of the dual of C wrt to the dual of $\leq$

Weighted (Co)Limits: Inserters and Coinserters

Weighted limits are the appropriate notion of limit for enriched category theory

Priestley duality lives in order enriched category theory

Order enriched categories have partially ordered homsets

A (co)inserter is the ordered version of a (co)equaliser

Let $\mathbf{Pos}$ be the category of partially ordered sets

The **inserter** of $f, g : X \to Y$ is the sub-poset of $X \times Y$ given by

$$\{(x, y) \mid fx \leq gy\}$$

The **coinserter** of $f, g : X \to Y$ is the posetal quotient of

$$(\ Y\ ,\ \leq_Y \cup \{(fx, gx) \mid x \in X\}\ )$$

Remark: Weighted limits have a definition by universal property that works in abstract categories, but we don't need to know it for this talk

Example: Beyond Zero-Dimensional Duality

How to extend the Stone/BA duality to ordered compact Hausdorff spaces?

Represent ordered compact Hausdorff spaces as $(X, \sqsubseteq)$ with X a Stone space and $\sqsubseteq$ a closed preorder

Idea: Dualise X and $\sqsubseteq$ separately:

There is a dual equivalence $2^{-} : \mathbf{Stone} \to \mathbf{BA}$

Extend this functor to $\overline{2} : \mathbf{Rel(Stone)} \to \mathbf{Rel(BA)}$

Then the dual of $(X, \sqsubseteq)$ can be represented as $(2^X, \overline{2}(\sqsubseteq))$

This observation can be developed into a general theory extending zero-dimensional dualities to continuous dualities (not in this talk)

Similarly:

Stone spaces with Stone-relations are dual to BAs with DL-relations

Ordered Stone spaces dual to BAs with interpolative relations below the order

The Dual of a Relation in the Case of Homming into $\mathbb{2}$

Let $\mathbb{2}^- : \mathcal{X} \to \mathcal{A}$ be, for example, one of the functors

$\mathbb{2}^- : \mathbf{Pos} \to \mathbf{Pos}$

$\mathbb{2}^- : \mathbf{Stone} \to \mathbf{BA}$ $\qquad$ $\mathbb{2}^- : \mathbf{BA} \to \mathbf{Stone}$

$\mathbb{2}^- : \mathbf{Pri} \to \mathbf{DL}$ $\qquad$ $\mathbb{2}^- : \mathbf{DL} \to \mathbf{Pri}$

The extension to binary relations is a functor

$$\overline{\mathbb{2}} : \mathbf{Rel}(\mathcal{X}) \to \mathbf{Rel}(\mathcal{A})$$
$$R \mapsto \{ (a, b) \mid R[a] \subseteq b \}$$

We will see later why $\overline{\mathbb{2}}$ is an equivalence of categories whenever $\mathbb{2}^-$ is

Example: Stralka's Ersatzkette

Recall

every ordered compact Hausdorff space is the coinserter of a Stone space

every Priesteley space $(X, \leq)$ is the coinserter of X by $\leq$

the dual $\mathbb{DL}$ is the inserter of $\mathbb{2}^X$ by $\overline{\mathbb{2}}(\leq)$

What if we start with an ordered Stone space that is not a Priestley space?

Cantor Space C : Middle-third-subset of the unit interval

Partial order $\leq$ linking the left to the right endpoint of each gap

$(a, b) \in \overline{\mathbb{2}}(\leq)$ iff $\leq[a] \subseteq b$

We can give a new argument why $(C, \leq)$ is not a Priestley space: The distributive lattice of reflexive elements of $\overline{\mathbb{2}}(\leq)$ is the two element lattice, which is not dual to $(C, \leq)$

Stralka (1980)

Example: d-Frames

Bitopological spaces (X, τ_-, τ_+)

The duals are d-frames (L_-, L_+, con, tot)

$$con : L_- \looparrowright L_+^{\mathsf{op}} \qquad tot : L_+^{\mathsf{op}} \looparrowright L_-$$

The dual of a d-frame consists of points $(p_+ : L_+ \to \mathbb{2}, p_- : L_- \to \mathbb{2})$

$$\forall (a_-, a_+) \in con \,.\, p_-(a_-) = 0 \text{ or } p_+(a_+) = 0$$
$$\forall (a_+, a_-) \in tot \,.\, p_-(a_-) = 1 \text{ or } p_+(a_+) = 1$$

Prop: The carrier of the dual of (L_-, L_+, con, tot) is $\overline{\mathbb{2}}(con) \cap \overline{\mathbb{2}}(tot)$.

Example: Hofmann-Mislove-Stralka Duality

What happens if one drops $\vee$ from distributive lattices?

On the algebra side: Meet Semi-Lattices

On the topological side: Hofmann-Mislove-Stralka spaces

HMS-spaces are MSLs in $\mathbf{Pri}$ where the order agrees with the MSL-order

Example: Banaschewski Duality

What happens if one drops $\vee$ and $\wedge$ from distributive lattices?

On the algebra side: Posets

On the topological side: Banaschewski spaces

B.-spaces are DLs in $\mathbf{Pri}$ where the order agrees with the DL-order

Example: The Self-Adjunction of $\mathbf{Pos}$ and $\mathbf{Set}$

Our construction works also for adjunctions, for example

$$\mathbf{Pos} \underset{\mathbb{2}^-}{\overset{\mathbb{2}^-}{\rightleftarrows}} \mathbf{Pos}$$

For example, let X be the integers and R an equivalence relation

The dual $\overline{\mathbb{2}}(R)$ relates sets (A, B) if $R[A] \subseteq B$

The reflexive elements $R[A] \subseteq A$ are unions of equivalence classes

The dual of the quotient X/R is given by the reflexive elements of the dual of R

The dual of the coinserter of X wrt R is the inserter of $\mathbb{2}^X$ wrt $\overline{\mathbb{2}}(R)$

Main Theorem 1

If

$U : \mathcal{X} \to \mathbf{Pos}$ and $V : \mathcal{A} \to \mathbf{Pos}$ are conretely-order regular categories

$F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ are a dual equivalence

preserving exact squares

Then

F and G extend to an equivalence of categories of relations

$$(\mathbf{Rel}\mathcal{X})^{\mathrm{co}} \underset{\mathbf{Rel}G}{\overset{\mathbf{Rel}F}{\rightleftarrows}} \mathbf{Rel}\mathcal{A}$$

Main Theorem 2

If

$U : \mathcal{X} \to \mathbf{Pos}$ and $V : \mathcal{A} \to \mathbf{Pos}$ are conretely-order regular categories

$F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ are a dual adjunction

preserving exact squares and mapping surjections to embeddings

Then

F and G extend to an adjunction of framed bicategories of relations

$$(\S\mathcal{X})^{\mathsf{co}} \underset{\S G}{\overset{\S F}{\rightleftarrows}} \S\mathcal{A}$$

Relations

Let $U : \mathcal{C} \to \mathbf{Pos}$ be a category (with some good properties ...)

Definition: A **relation** $R : A \looparrowright B$ in $\mathcal{C}$ is a

- sub-object $R \subseteq A \times B$ that is also
- an order-preserving map $A^{\mathsf{op}} \times B \to \mathbb{2}$

where $\mathbb{2} = \{0 < 1\}$

Remark: Also called *monotone* or *weakening (closed)* relations

Examples: Stone-relations, BA-relations, Priestley-relations, DL-relations, ...

Relations

Let $U : \mathcal{C} \to \mathbf{Pos}$ be a category (with some good properties ...)

Definition: A **relation** $R : A \looparrowright B$ in $\mathcal{C}$ is a

- sub-object $R \subseteq A \times B$ that is also

- an order-preserving map $A^{\mathsf{op}} \times B \to \mathbb{2}$

where $\mathbb{2} = \{0 < 1\}$

Remark: Also called *monotone* or *weakening (closed)* relations

Examples: Stone-relations, BA-relations, Priestley-relations, DL-relations, ...

Relations

Let $U : \mathcal{C} \to \mathbf{Pos}$ be a category (with some good properties ...)

Definition: A **relation** $R : A \looparrowright B$ in $\mathcal{C}$ is a

- sub-object $R \subseteq A \times B$ that is also

- an order-preserving map $A^{\mathsf{op}} \times B \to \mathbb{2}$

where $\mathbb{2} = \{0 < 1\}$

Remark: Also called *monotone* or *weakening (closed)* relations

Examples: Stone-relations, BA-relations, Priestley-relations, DL-relations, ...

Motivating Example 2: Sequent Calculi

Our first motivating example started on the topological side, quotienting zero dimensional Stone spaces to continuous Hausdorff spaces ... but we can also begin on the algebraic/logical side

Relations in $\mathbf{DL}$ (distributive lattices) are essentially sequent cacluli:

Taking subobjects in $\mathbf{DL}$ amounts to

$$\frac{}{0R0} \qquad \frac{}{1R1} \qquad \frac{aRb \quad a'Rb'}{(a \wedge a')R(b \wedge b')} \qquad \frac{aRb \quad a'Rb'}{(a \vee a')R(b \vee b')}$$

while the monotonicity condition amounts to weakening

$$\frac{a' \leq a \quad aRb \quad b \leq b'}{a'Rb'}$$

“An $\mathcal{A}$-relation for a category $\mathcal{A}$ of ordered algebras is a sequent calculus”

Heterogeneous Relations and Mixed Variance

The topological story proceeds by splitting of idempotents, which generates *heterogeneous relations* (= relations as arrows $A \nrightarrow B$ with $A \neq B$)

While heterogeneous relations work nicely also on the algebraic side, it is not clear how to extend our methodology to allow for mixed variance in the presence of heterogeneous relations

Indeed, rules such as

$$\frac{a\,R\,b}{\neg b\,R\,\neg a} \qquad\qquad \frac{a_1\,R\,b_1 \quad a_2\,R\,b_2}{(b_1 \to a_2)\,R\,(a_1 \to b_2)}$$

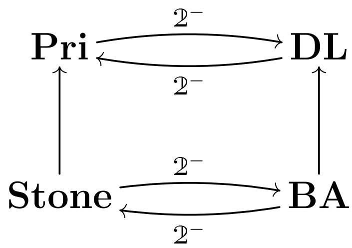
only make sense if the a and the b are drawn from the same set, that is, for relations $R : A \nrightarrow B$ with $A = B$

To summarise, in this work, we only consider algebraic signatures in which **all operations are monotone**

Order Enriched Stone Duality

It is important to understand that 'all operations are monotone' does not exclude the category $\mathbf{BA}$ of Boolean algebras

We simply take $\mathbf{BA}$ as a full subcategory of $\mathbf{DL}$



This makes the order enrichment visible for $\mathbf{Stone}$ and $\mathbf{BA}$

In particular, the dual of a Stone-relation R is given by

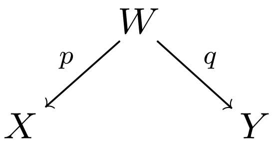
$$(a, b) \in \mathbb{2}(R) \;\Leftrightarrow\; R[a] \subseteq b$$

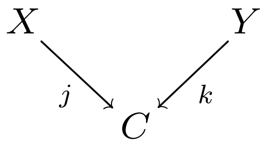
This relies on the dual of a Stone space being ordered by $\subseteq$

Relations as Spans and Cospans

Relations can be tabulated as

spans and as *cospans*





with

$$xRy \Leftrightarrow \exists w \,.\, x \leq pw \;\&\; qw \leq y \qquad\qquad xRy \Leftrightarrow \; jx \leq_C ky$$

While for spans the $\leq$ is not essential, it is for cospans:

the order $\leq_C$ of C encodes the relation R

This is the point which is responsible for working *order enriched* and which does not generalise to lattices of truth values other than $\mathbb{2}$ in an obvious way (ie many-valued relations, see the conclusions)

Relations as Equivalence Classes of (Co)Spans

Different spans, and different cospans, can represent the same relation

For example, the span W and the cospan C may have redundant elements

Each equivalence class has a normal form (via Onto-Embedding factorisations)

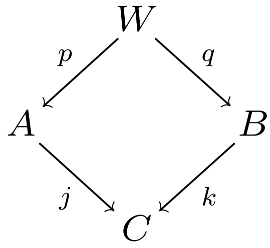
Alternatively, we can compute the span-normal form of a relation represented as a cospan as the 'ordered kernel' of the cospan

And we can compute the cospan-normal form of a relation represented as a span as the 'ordered pushout' of the span

These are the two other examples of weighted limits we need

Weighted (Co)Limits 2: Commas and Cocommas

In the category $\mathbf{Pos}$ the diagram



- is a **comma square** (and we also say that W is the comma of (j, k)) if

$$W = \{(a, b) \mid ja \leq kb\}$$

- is a **cocomma square** (and we also say that C is the cocomma of (p, q)) if $C = (A + B)/\sqsubseteq$ where $a \sqsubseteq b \iff \exists w \,.\, a \leq_A pw \;\&\; qw \leq_B b$

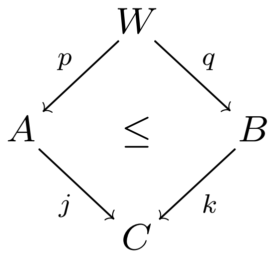
Intuitively, the comma is the *graph* of the relation and the cocomma is the *order-quotient* of the relation

Proposition: Two spans represent the same relation iff they have isomorphic cocommas. Two cospans are equivalent iff they have isomorphic commas

Exact Squares

Exact squares were introduced by Hilton in the context of abelian categories and generalised by Guitart to 2-categories. We apply these ideas to order enriched categories

A diagram in $\mathbf{Pos}$



is called **exact** if $Rel(p, q) = Rel(j.k)$.

Proposition: Comma and cocomma squares in $\mathbf{Pos}$ are exact

$\mathbf{Rel}(\mathbf{Pos})$ is the ordered category of spans (or cospans) modulo exact squares

Concretely Order-Regular Categories

Aim: Generalise $\mathbf{Rel}(\mathbf{Pos})$ to $\mathbf{Rel}(\mathcal{C})$ for suitable categories $\mathcal{C}$

In concretely-order regular categories relations behave as in $\mathbf{Pos}$

Definition: $U : \mathcal{C} \to \mathbf{Pos}$ is concretely-order regular if

- U is order faithful (injective and order-reflecting on homsets)
- $\mathcal{C}$ has and U preserves finite weighted limits
- $\mathcal{C}$ has and U preserves Onto-Embedding factorisations

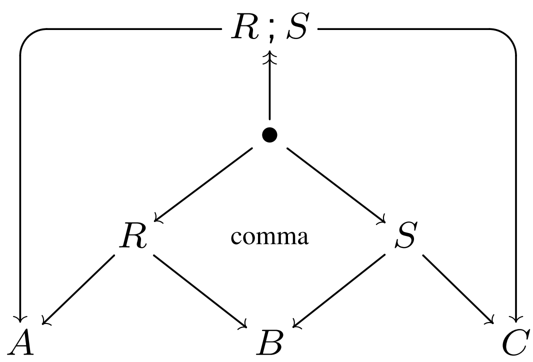
The last item can be replaced by "existence of exact cocommas" and the last two items can be replaced by "existence of enough exact squares"

The Category $\mathbf{Rel}\mathcal{C}$ of Relations in $\mathcal{C}$

Relations are equivalence classes of weakening closed spans

Each equivalence class has a canonical represenatative (up to iso) obtained from Onto-Embedding factorisation

Composition of relations R and S is via comma squares and factorisation:



Main Theorem 1

If

$U : \mathcal{X} \to \mathbf{Pos}$ and $V : \mathcal{A} \to \mathbf{Pos}$ are conretely-order regular categories

$F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ are a dual equivalence

preserving exact squares

Then

F and G extend to an equivalence of categories of relations

$$(\mathbf{Rel}\mathcal{X})^{\mathsf{co}} \underset{\mathbf{Rel}G}{\overset{\mathbf{Rel}F}{\rightleftarrows}} \mathbf{Rel}\mathcal{A}$$

Sketch of Proof of Theorem 1

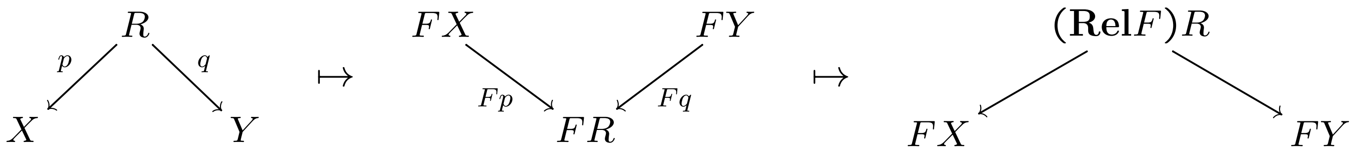
Extending functors to relations for ordinary categories goes back to Barr (1970)

Adapted to the order enriched setting by Bilkova-Kurz-Petrisan-Velebil (2012)

Let $F : \mathcal{C} \to \mathcal{C}$ be a functor

Tabulate a relation as a span (p, q) and map the span to (Fp, Fq)

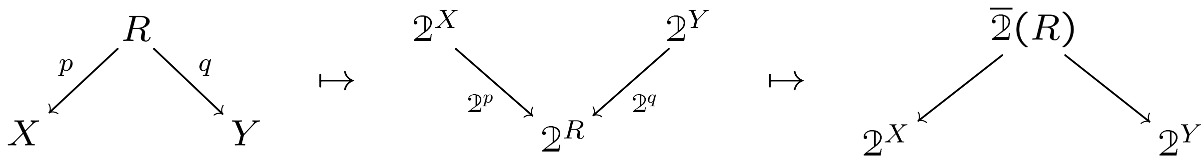
In the theorem, F is contravariant, so we go via a cospan to a comma



Example: Homming into $\mathbb{2}$

Let $\overline{\mathbb{2}}$ be the extension of the contravariant ‘upper set functor’ $\mathbb{2}^{-} : \mathbf{Pos} \to \mathbf{Pos}$

$\overline{\mathbb{2}}$ acts on relations as follows



For all upper sets $a \subseteq X$, $b \subseteq Y$

$$\begin{aligned} (a,b) \in \overline{\mathbb{2}}(R) &\iff \mathbb{2}^{p}(a) \subseteq \mathbb{2}^{q}(b) \\ &\iff \forall x \in a\,.\forall y \in Y.\, xRy \Rightarrow y \in b \\ &\iff R[a] \subseteq b \end{aligned}$$

Main Theorem 2

In the theorem below, we cannot replace $\S\mathcal{X}, \S\mathcal{A}$ by $\mathbf{Rel}\mathcal{X}, \mathbf{Rel}\mathcal{A}$ because the unit and the counit of the extended adjunction are only natural wrt to maps, not wrt relations

If

$U : \mathcal{X} \to \mathbf{Pos}$ and $V : \mathcal{A} \to \mathbf{Pos}$ are conretely-order regular categories

$F : \mathcal{X} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{X}$ are a dual adjunction

preserving exact squares and mapping surjections to embeddings

Then

F and G extend to an adjunction of framed bicategories of relations

$$(\S\mathcal{X})^{\mathrm{co}} \underset{\S G}{\overset{\S F}{\rightleftarrows}} \S\mathcal{A}$$

Framed Bicategories

Shulman's framed bicategories are particular double categories in which the 'vertical' arrows behave like maps and the 'horizontal' arrows like relations

Framed bicategories organise themselves in a 2-category

2-categories come with a native notion of adjunction

Spelling out the details, one finds that this notion of adjunction requires naturality only wrt to vertical arrows (maps)

References, Background, Related Work

Nachbin: Topology and Order (1965)

Barr: Relational algebras (1970)

Priestley: Representation of Distributive Lattices (1970)

Scott: Continuous lattices (1972)

Lawvere: Metric spaces, generalized logic and closed categories (1973)

Guitart: Relations et carrés exacts (1980)

Street: Fibrations in bicategories (1980)

Kelly: Basic Concepts of Enriched Category Theory (1982)

Smyth, Plotkin: The C.-T. Solution of Recursive Domain Equations (1982)

Johnstone, Stone Spaces (1982)

Abramsky: Domain Theory in Logical Form (1991)

Jung, Kegelmann, Moshier: Multi Lingual Sequent Calculus (1999)

Shulman: Framed bicategories and monoidal fibrations (2008)

Bilkova, Kurz, Petrisan, Velebil: Relation Liftings on Posets (2012,2013)

Conclusion

Extend Stone duality from maps to relations

In preparation: Extending zero-dimensional dualities to continuous dualities

What we have done:

- category theory of *cat's enriched over cat's enriched over* $\mathbb{2}$
- examples with *dualising object* $\mathbb{2}$

Future work:

- (more of the above)
- *many-valued valuations*: general dualising poset of truth values (replacing $\mathbb{2}$)
- *many-valued relations*: enrich over lattice of truth values (replacing $\mathbb{2}$)

Other dualising objects could lead to new results for many-valued logic?

Ask me for a preprint if you are interested ...