BITOPOLOGICAL DUALITY FOR DISTRIBUTIVE LATTICES AND HEYTING ALGEBRAS

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Abstract. We introduce pairwise Stone spaces as a natural bitopological generalization of Stone spaces—the duals of Boolean algebras—and show that they are exactly the bitopological duals of bounded distributive lattices. The category **PStone** of pairwise Stone spaces is isomorphic to the category **Spec** of spectral spaces and to the category **Pries** of Priestley spaces. In fact, the isomorphism of **Spec** and **Pries** is most naturally seen through **PStone** by first establishing that **Pries** is isomorphic to **PStone**, and then showing that **PStone** is isomorphic to **Spec**. We provide the bitopological and spectral descriptions of many algebraic concepts important for the study of distributive lattices. We also give new bitopological and spectral dualities for Heyting algebras, thus providing two new alternatives to Esakia's duality.

1. Introduction

It is widely considered that the beginning of duality theory was Stone's groundbreaking work in the mid 30s on the dual equivalence of the category **Bool** of Boolean algebras and Boolean algebra homomorphism and the category **Stone** of compact Hausdorff zero-dimensional spaces, which became known as Stone spaces, and continuous functions. In 1937 Stone [33] extended this to the dual equivalence of the category **DLat** of bounded distributive lattices and bounded lattice homomorphisms and the category **Spec** of what later became known as spectral spaces and spectral maps. Spectral spaces provide a generalization of Stone spaces. Unlike Stone spaces, spectral spaces are not Hausdorff (not even T_1)[1], and as a result, are more difficult to work with. In 1970 Priestley [25] described another dual category of **DLat** by means of special ordered Stone spaces, which became known as Priestley spaces, thus establishing that **DLat** is also dually equivalent to the category **Pries** of Priestley spaces and continuous order-preserving maps. Since **DLat** is dually equivalent to both **Spec** and **Pries**, it follows that the categories **Spec** and **Pries** are equivalent. In fact, more is true: as shown by Cornish [6] (see also Fleisher [11]), **Spec** is actually isomorphic to **Pries**.

Spectral spaces are more natural to work with from the point of view of pointfree topology, as demonstrated by Johnstone [17]. In addition, spectral spaces only have a topological structure, while Priestley spaces also have an order structure on top of topology, thus their signature is more complicated than that of spectral spaces. However, Priestley spaces arise more naturally in relation with logics, as Priestley spaces incorporate the now widely used Kripke semantics in them. As a result, Priestley's duality became rather popular among logicians, and most dualities for distributive lattices with operators have been performed in

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[1]In fact, a spectral space X is a Stone space iff X is T_1.

terms of Priestley spaces. Here we only mention Esakia's duality for Heyting algebras [8], which is a restricted version of Priestley's duality.[2]

Another way to represent distributive lattices is by means of bitopological spaces, as demonstrated by Jung and Moshier [20]. In fact, bitopological spaces provide a natural medium in establishing the isomorphism between **Pries** and **Spec**: with each Priestley space $(X, \tau, \leq)$, there are two natural topologies associated with it; the upper topology τ_1 consisting of open upsets of $(X, \tau, \leq)$, and the lower topology τ_2 consisting of open downsets of $(X, \tau, \leq)$. Consequently, (X, τ_1, τ_2) is a bitopological space. Moreover, both topologies τ_1 and τ_2 are spectral topologies, the Priestley topology τ is in fact the join of τ_1 and τ_2, and the spectral space associated with $(X, \tau, \leq)$ is obtained from (X, τ_1, τ_2) by simply forgetting τ_2.

In this paper we provide an explicit axiomatization of the class of bitopological spaces obtained this way. We call these spaces *pairwise Stone spaces*. On the one hand, pairwise Stone spaces provide a natural generalization of Stone spaces as each of the three conditions defining a Stone space naturally generalizes to the bitopological setting: compact becomes pairwise compact, Hausdorff – pairwise Hausdorff, and zero-dimensional – pairwise zero-dimensional. On the other hand, pairwise Stone spaces provide a natural medium in moving from Priestley spaces to spectral spaces and backwards, thus Cornish's isomorphism of **Pries** and **Spec** can be established more naturally by first showing that **Pries** is isomorphic to the category **PStone** of pairwise Stone spaces and bicontinuous maps, and then showing that **PStone** is isomorphic to **Spec**. Thirdly, the signature of pairwise Stone spaces naturally carries the symmetry present in Priestley spaces (and distributive lattices), but hidden in spectral spaces. Moreover, the proof that **DLat** is dually equivalent to **PStone** is simpler than the existing proofs of the dual equivalence of **DLat** with **Spec** and **Pries**. Lastly, the isomorphism of **Pries**, **PStone**, and **Spec** fits nicely in a more general isomorphism of the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces described in [14, Ch. VI-6] (see also [30] and [24]).

The dualities described above have many applications in logic and computer science. In fact, the basic idea underlying completeness results of (propositional) logics is based on duality theory as the canonical model of a propositional logic is the dual of the Lindenbaum-Tarski algebra of the logic. Duality theory also provides a framework for understanding the relationship between denotational semantics of programs and program logics. In particular, as was shown by Abramsky [1], the denotational semantics and the corresponding program logic are duals of each other. For a recent application of these ideas to the π-calculus see [4]. For an application of duality theory to regular languages we refer to Gehrke et al. [12]. For a variety of applications of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces in probabilistic systems, we refer to the work of Jung, Moshier, and their collaborators [18, 19, 3, 20]. Here we only mention that there is a dual equivalence between these categories and the category of proximity lattices [32, 21], which are a generalization of distributive lattices, thus providing an interesting generalization of the duality for distributive lattices. We view our pairwise Stone spaces as a particular case of pairwise compact pairwise regular bitopological spaces, and our isomorphism of the categories of Priestley spaces, pairwise Stone spaces, and spectral spaces as a particular case

[2]We note that Esakia's work was independent of Priestley's; a proof that Esakia spaces are Priestley spaces can be found in [10, p. 62].

of the isomorphism of the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces.

One of the advantages of Priestley's duality is that many algebraic concepts important for the study of distributive lattices can be easily described by means of Priestley spaces. In addition, we show that they have a natural dual description by means of pairwise Stone spaces. We also give their dual description by means of spectral spaces, which at times is less transparent than the order topological and bitopological descriptions.

Finally, we introduce the subcategories of **PStone** and **Spec**, which are isomorphic to the category **Esa** of Esakia spaces and dually equivalent to the category **Heyt** of Heyting algebras. This provides an alternative to Esakia's duality in the setting of bitopological spaces and spectral spaces.

The paper is organized as follows. In Section 2 we recall some basic facts about bitopological spaces, introduce pairwise Stone spaces, and study their basic properties. In Section 3 we prove that the category **PStone** of pairwise Stone spaces is isomorphic to the category **Pries** of Priestley spaces. In Section 4 we prove that **PStone** is isomorphic to the category **Spec** of spectral spaces, thus establishing that all three categories are isomorphic to each other. In Section 5 we give a direct proof that the category **DLat** of distributive lattices is dually equivalent to **PStone**, thus providing an alternative of Stone's and Priestley's dualities. In Section 6 we give the dual description of many algebraic concepts important for the study of distributive lattices by means of Priestley spaces, pairwise Stone spaces, and spectral spaces. In particular, we give the dual description of filters, prime filters, maximal filters, ideals, prime ideals, maximal ideals, homomorphic images, sublattices, complete lattices, McNeille completions, and canonical completions. At the end of the section we list all the obtained results in one table, which can be viewed as a dictionary of duality theory for distributive lattices, complementing the dictionary given in [27]. Finally, in Section 7 we develop new bitopological and spectral dualities for Heyting algebras, thus providing an alternative to Esakia's duality, and give a table similar to the one given at the end of Section 6, which can be viewed as a dictionary of duality theory for Heyting algebras.

2. Pairwise Stone spaces

We recall that a *bitopological space* is a triple (X, τ_1, τ_2), where X is a nonempty set and τ_1 and τ_2 are two topologies on X. Ever since Kelly [22] introduced them, bitopological spaces have been subject of intensive investigation of many topologists. In particular, there has been a lot of research on the "correct" generalization of the basic topological properties to the bitopological setting. A large number of results obtained in this direction is collected in a recent monograph [7]. For our purposes it is important to find the right generalization of the concept of a Stone space. Therefore, we are interested in the bitopological versions of compactness, Hausdorffness, and zero-dimensionality.

There are several ways to generalize a topological property to the bitopological setting. Let (X, τ_1, τ_2) be a bitopological space and let $\tau = \tau_1 \vee \tau_2$. For a topological property P, we say that (X, τ_1, τ_2) is *bi-P* if both (X, τ_1) and (X, τ_2) are P, and we say that (X, τ_1, τ_2) is *join P* if (X, τ) is P. For example, (X, τ_1, τ_2) is bi-T_0, bi-T_1, or bi-T_2 if both (X, τ_1) and (X, τ_2) are T_0, T_1, or T_2, respectively; and (X, τ_1, τ_2) is join T_0, join T_1, or join T_2 if (X, τ) is T_0, T_1, or T_2, respectively. However, for our purposes, neither bi-Stone nor join Stone turns out to be the right generalization of the concept of a Stone space to the bitopological setting.

Definition 2.1. *Let (X, τ_1, τ_2) be a bitopological space.*

(1) [29, Def. 2.1.1] *We call* (X, τ_1, τ_2) pairwise T_0 *if for any two distinct points* $x, y \in X$ *there exists* $U \in \tau_1 \cup \tau_2$ *containing exactly one of* x, y.
(2) [29, Def. 2.1.3] *We call* (X, τ_1, τ_2) pairwise T_1 *if for any two distinct points* $x, y \in X$ *there exists* $U \in \tau_1 \cup \tau_2$ *such that* $x \in U$ *and* $y \notin U$.
(3) [29, Def. 2.1.8] *We call* (X, τ_1, τ_2) pairwise T_2 *or* pairwise Hausdorff *if for any two distinct points* $x, y \in X$ *there exist disjoint* $U \in \tau_1$ *and* $V \in \tau_2$ *such that* $x \in U$ *and* $y \in V$ *or there exist disjoint* $U \in \tau_2$ *and* $V \in \tau_1$ *with the same property.*

Remark 2.2. We have chosen [29] as our primary source of reference, although the concepts of a pairwise T_0 space and a pairwise T_1 space have appeared earlier in the literature.

Remark 2.3. It would be more in the vein of Definition 2.1(1) and 2.1(2) if we defined a pairwise T_2 space as a bitopological space satisfying the following condition: For any two distinct points $x, y \in X$ there exist disjoint $U, V \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$. Obviously if (X, τ_1, τ_2) is pairwise T_2, then it satisfies the condition above, but the converse is not true in general. Nevertheless, we will show below that in the realm of pairwise zero-dimensional spaces the two conditions are equivalent.

For a bitopological space (X, τ_1, τ_2), let δ_1 denote the collection of closed subsets of (X, τ_1) and δ_2 denote the collection of closed subsets of (X, τ_2). The next definition generalizes the notion of zero-dimensionality to bitopological spaces.

Definition 2.4. [28, p. 127] *We call a bitopological space* (X, τ_1, τ_2) pairwise zero-dimensional *if opens in* (X, τ_1) *closed in* (X, τ_2) *form a basis for* (X, τ_1) *and opens in* (X, τ_2) *closed in* (X, τ_1) *form a basis for* (X, τ_2)*; that is,* $\beta_1 = \tau_1 \cap \delta_2$ *is a basis for* τ_1 *and* $\beta_2 = \tau_2 \cap \delta_1$ *is a basis for* τ_2.

We point out that if (X, τ_1, τ_2) is pairwise zero-dimensional, then $\beta_2 = \{U^c \mid U \in \beta_1\}$ and $\beta_1 = \{V^c \mid V \in \beta_2\}$. Moreover, both β_1 and β_2 contain $\emptyset, X$ and are closed with respect to finite unions and intersections.

Lemma 2.5. *Suppose that* (X, τ_1, τ_2) *is pairwise zero-dimensional. Then the following conditions are equivalent:*

(1) (X, τ_1) *is* T_0.
(2) (X, τ_2) *is* T_0.
(3) (X, τ_1, τ_2) *is pairwise* T_2.
(4) *For any two distinct points* $x, y \in X$ *there exist disjoint* $U, V \in \tau_1 \cup \tau_2$ *such that* $x \in U$ *and* $y \in V$.
(5) (X, τ_1, τ_2) *is join* T_2.
(6) (X, τ_1, τ_2) *is bi-*T_0.

Proof. (1)⇒(2): Suppose that (X, τ_1) is T_0 and x, y are two distinct points of X. Then there exists $U \in \tau_1$ containing exactly one of x, y. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since (X, τ_1, τ_2) is pairwise zero-dimensional, there exists $V \in \beta_1$ such that $x \in V \subseteq U$. Therefore, $V^c \in \beta_2$, $y \in V^c$, and $x \notin V^c$. Thus, (X, τ_2) is T_0.

(2)⇒(3): Suppose that (X, τ_2) is T_0 and x, y are two distinct points of X. Then there exists $U \in \tau_2$ containing exactly one of x, y. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since (X, τ_1, τ_2) is pairwise zero-dimensional, there exists $V \in \beta_2$ such that $x \in V \subseteq U$. Then $x \in V \in \beta_2$, $y \in V^c \in \beta_1$, and V, V^c are disjoint. Thus, (X, τ_1, τ_2) is pairwise T_2.

(3)⇒(4) is obvious.

(4)⇒(5): Suppose that x, y are two distinct points of X. By (4), there exist disjoint $U, V \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$. Without loss of generality we may assume that $U, V \in \tau_1$. Since (X, τ_1, τ_2) is pairwise zero-dimensional, there exists $U' \in \beta_1$ such that $x \in U' \subseteq U$. Let $V' = X - U'$. Then $V \subseteq V'$, so $y \in V' \in \beta_2$, and so there exist two disjoint τ-open sets U', V' such that $x \in U'$ and $y \in V'$. Thus, (X, τ_1, τ_2) is join T_2.

(5)⇒(6): Suppose that (X, τ_1, τ_2) is join T_2. We show that (X, τ_1) is T_0. Let x, y be two distinct points of X. Since (X, τ_1, τ_2) is pairwise zero-dimensional and join T_2, there exist $U_1, U_2 \in \beta_1$ and $V_1, V_2 \in \beta_2$ such that $x \in U_1 \cap V_1$, $y \in U_2 \cap V_2$, and $U_1 \cap V_1$ and $U_2 \cap V_2$ are disjoint. If $y \notin U_1$, then there is $U_1 \in \tau_1$ containing exactly one of x, y. If $y \in U_1$, then $y \notin V_1$. Therefore, $y \in U_2 \cap V_1^c$. Clearly $U_2 \cap V_1^c \in \beta_1$. Moreover, $x \notin U_2 \cap V_1^c$ as $x \notin V_1^c$. Thus, there exists $U_2 \cap V_1^c \in \tau_1$ containing exactly one of x, y. In either case, we separate x, y by a τ_1-open set, and so (X, τ_1) is T_0. That (X, τ_2) is T_0 is proved similarly. Consequently, (X, τ_1, τ_2) is bi-T_0.

(6)⇒(1) is obvious. ⊣

On the other hand, (X, τ_1, τ_2) may be pairwise zero-dimensional and pairwise T_2 without either of τ_1, τ_2 being even T_1 as the following simple example shows.

Example 2.6. Let $X = \{0, 1\}$, $\tau_1 = \{\emptyset, \{1\}, X\}$ and $\tau_2 = \{\emptyset, \{0\}, X\}$. Then both τ_1 and τ_2 are the Sierpinski topologies on X, thus both are T_0, but not T_1. Nevertheless, (X, τ_1, τ_2) is pairwise zero-dimensional and pairwise T_2.

The next definition generalizes the notion of compactness to bitopological spaces.

Definition 2.7. [29, Def. 2.2.17] *We call a bitopological space* (X, τ_1, τ_2) pairwise compact *if for each cover* $\{U_i \mid i \in I\}$ *of* X *with* $U_i \in \tau_1 \cup \tau_2$*, there exists a finite subcover.*

Remark 2.8. In [29, Def. 2.2.17] Salbany defines a bitopological space (X, τ_1, τ_2) to be pairwise compact if (X, τ) is compact, where $\tau = \tau_1 \vee \tau_2$. In our terminology this means that (X, τ_1, τ_2) is join compact. But it is a consequence of Alexander's Lemma—a classical result in general topology—that the two notions of pairwise compact and join compact coincide.

It is obvious that if (X, τ_1, τ_2) is pairwise compact, then both (X, τ_1) and (X, τ_2) are compact; that is, (X, τ_1, τ_2) is bi-compact. On the other hand, it was observed by Salbany [29, p. 17] that the converse is not true in general. Let σ_1 and σ_2 denote the collections of compact subsets of (X, τ_1) and (X, τ_2), respectively.

Proposition 2.9. *A bitopological space* (X, τ_1, τ_2) *is pairwise compact iff* $\delta_1 \subseteq \sigma_2$ *and* $\delta_2 \subseteq \sigma_1$.

Proof. [⇒] Suppose that (X, τ_1, τ_2) is pairwise compact. We show that $\delta_1 \subseteq \sigma_2$. Let $A \in \delta_1$ and let $A \subseteq \bigcup\{U_i \mid i \in I\}$ with $\{U_i \mid i \in I\} \subseteq \tau_2$. Then the collection $\{U_i \mid i \in I\} \cup \{A^c\}$ is a cover of X. Since $A^c \in \tau_1$ and (X, τ_1, τ_2) is pairwise compact, there exist $i_1, \ldots, i_n \in I$ such that $U_{i_1} \cup \cdots \cup U_{i_n} \cup A^c = X$. It follows that $A \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$, and so $A \in \sigma_2$. Thus, $\delta_1 \subseteq \sigma_2$. That $\delta_2 \subseteq \sigma_1$ is proved similarly.

[⇐] Suppose that $\delta_1 \subseteq \sigma_2$ and $\delta_2 \subseteq \sigma_1$. To show that (X, τ_1, τ_2) is pairwise compact let $\{U_i \mid i \in I\} \subseteq \tau_1$ and $\{V_j \mid j \in J\} \subseteq \tau_2$ with $\bigcup\{U_i \mid i \in I\} \cup \bigcup\{V_j \mid j \in J\} = X$. We set $U = \bigcup\{U_i \mid i \in I\}$. Clearly $U \in \tau_1$ and $U \cup \bigcup\{V_j \mid j \in J\} = X$, so $U^c \subseteq \bigcup\{V_j \mid j \in J\}$. Since $U^c \in \delta_1$ and $\delta_1 \subseteq \sigma_2$, we have that $U^c \in \sigma_2$. Therefore, there exist $j_1, \ldots, j_n \in J$ such that $U^c \subseteq V_{j_1} \cup \cdots \cup V_{j_n}$. We set $V = V_{j_1} \cup \cdots \cup V_{j_n}$. Then $U \cup V = X$, so $V^c \subseteq U = \bigcup\{U_i \mid i \in I\}$.

Since $V^c \in \delta_2$ and $\delta_2 \subseteq \sigma_1$, we have that $V^c \in \sigma_1$. Therefore, there exist $i_1, \dots, i_m \in I$ such that $V^c \subseteq U_{i_1} \cup \dots \cup U_{i_m}$. Clearly the finite collection $\{V_{j_1}, \dots, V_{j_n}, U_{i_1}, \dots, U_{i_m}\}$ is a cover of X. Thus, X is pairwise compact. $\dashv$

Now we generalize the notion of a Stone space to that of a pairwise Stone space.

Definition 2.10. *We call* (X, τ_1, τ_2) *a* pairwise Stone space *if it is pairwise compact, pairwise Hausdorff, and pairwise zero-dimensional.*

Remark 2.11. In the definition of a pairwise Stone space, pairwise Hausdorff can be replaced by any of the equivalent conditions of Lemma 2.5, and that pairwise compact can be replaced by $\delta_1 \subseteq \sigma_2$ and $\delta_2 \subseteq \sigma_1$, as follows from Proposition 2.9.

Let **PStone** denote the category of pairwise Stone spaces and bi-continuous functions; that is functions which are continuous with respect to both topologies.

3. Priestley spaces and pairwise Stone spaces

Let $(X, \leq)$ be a poset. We recall that $A \subseteq X$ is an *upset* if $x \in A$ and $x \leq y$ imply $y \in A$, and that A is a *downset* if $x \in A$ and $y \leq x$ imply $y \in A$. For $Y \subseteq X$ let $\uparrow Y = \{x \mid \exists y \in Y \text{ with } y \leq x\}$ and $\downarrow Y = \{x \mid \exists y \in Y \text{ with } x \leq y\}$. Let $\mathsf{Up}(X)$ denote the set of upsets and $\mathsf{Do}(X)$ denote the set of downsets of $(X, \leq)$.

Let $(X, \tau, \leq)$ be an ordered topological space. We denote by $\mathsf{OpUp}(X)$ the set of open upsets, by $\mathsf{ClUp}(X)$ the set of closed upsets, and by $\mathsf{CpUp}(X)$ the set of clopen upsets of $(X, \tau, \leq)$. Similarly, let $\mathsf{OpDo}(X)$ denote the set of open downsets, $\mathsf{ClDo}(X)$ denote the set of closed downsets, and $\mathsf{CpDo}(X)$ denote the set of clopen downsets of $(X, \tau, \leq)$. The next definition is well-known.

Definition 3.1. *An ordered topological space* $(X, \tau, \leq)$ *is a* Priestley space *if* (X, τ) *is compact and whenever* $x \not\leq y$*, there exists a clopen upset* A *such that* $x \in A$ *and* $y \notin A$.

The second condition in the above definition is known as the *Priestley separation axiom* (PSA for short). The next lemma is well-known.

Lemma 3.2. *Let* $(X, \tau, \leq)$ *be an ordered topological space.*

1. *If* $(X, \tau, \leq)$ *is a Priestley space, then* (X, τ) *is a Stone space.*
2. *If* $(X, \tau, \leq)$ *is a Priestley space, then* $\uparrow F$ *and* $\downarrow F$ *are closed for each closed subset* F *of* X.
3. *In a Priestley space, every open upset is the union of clopen upsets, every closed upset is the intersection of clopen upsets, every open downset is the union of clopen downsets, and every closed downset is the intersection of clopen downsets.*
4. *In a Priestley space, clopen upsets and clopen downsets form a subbasis for the topology.*
5. $(X, \tau, \leq)$ *is a Priestley space iff* (X, τ) *is compact and for closed subsets* F *and* G *of* X*, whenever* $\uparrow F \cap \downarrow G = \emptyset$*, there exists a clopen upset* A *of* X *such that* $F \subseteq A$ *and* $G \subseteq A^c$.

We will refer to condition (5) in the lemma as the *strong Priestley separation axiom* (SPSA for short). Let **Pries** denote the category of Priestley spaces and continuous order-preserving maps. We show that the categories **Pries** and **PStone** are isomorphic. To this end, we will define two functors Φ : **PStone** $\to$ **Pries** and Ψ : **Pries** $\to$ **PStone** which will set the required isomorphism.

For a topological space (X,τ), let $\leq$ denote the *specialization order* of (X,τ); that is,

$$x \leq y \text{ iff } x \in \mathrm{Cl}(y) \text{ iff } (\forall U \in \tau)(x \in U \text{ implies } y \in U).$$

It is well-known that $\leq$ is reflexive and transitive, and that $\leq$ is antisymmetric iff (X,τ) is T_0.

Lemma 3.3. *Let (X,τ_1,τ_2) be a bitopological space, $\leq_1$ be the specialization order of (X,τ_1), and $\leq_2$ be the specialization order of (X,τ_2). If (X,τ_1,τ_2) is pairwise zero-dimensional, then $\leq_1=\geq_2$.*

Proof. Let (X,τ_1,τ_2) be pairwise zero-dimensional; that is, $\beta_1=\tau_1\cap\delta_2$ is a basis for τ_1 and $\beta_2=\tau_2\cap\delta_1$ is a basis for τ_2. Then, for each $x,y\in X$, we have:

$$\begin{array}{lll} x \leq_1 y & \text{iff} & (\forall U \in \tau_1)(x \in U \text{ implies } y \in U) \\ & \text{iff} & (\forall U \in \beta_1)(x \in U \text{ implies } y \in U) \\ & \text{iff} & (\forall U \in \beta_1)(y \in U^c \text{ implies } x \in U^c) \\ & \text{iff} & (\forall V \in \beta_2)(y \in V \text{ implies } x \in V) \\ & \text{iff} & (\forall V \in \tau_2)(y \in V \text{ implies } x \in V) \\ & \text{iff} & y \leq_2 x. \end{array}$$

⊣

For a pairwise Stone space (X,τ_1,τ_2), let $\tau=\tau_1\vee\tau_2$, and let $\leq=\leq_1$ be the specialization order of (X,τ_1).

Proposition 3.4. *If (X,τ_1,τ_2) is a pairwise Stone space, then $(X,\tau,\leq)$ is a Priestley space. Moreover:*

(i) $\mathsf{CpUp}(X,\tau,\leq)=\beta_1$.
(ii) $\mathsf{OpUp}(X,\tau,\leq)=\tau_1$.
(iii) $\mathsf{ClUp}(X,\tau,\leq)=\delta_2$.
(iv) $\mathsf{CpDo}(X,\tau,\leq)=\beta_2$.
(v) $\mathsf{OpDo}(X,\tau,\leq)=\tau_2$.
(vi) $\mathsf{ClDo}(X,\tau,\leq)=\delta_1$.

Proof. Since (X,τ_1,τ_2) is pairwise compact, (X,τ_1,τ_2) is join compact, and so (X,τ) is compact. Also, as (X,τ_1,τ_2) is pairwise Hausdorff, it follows from Lemma 2.5 that (X,τ_1) is T_0. Therefore, $\leq=\leq_1$ is a partial order. We show that $(X,\tau,\leq)$ satisfies PSA. If $x\not\leq y$, then $x\not\leq_1 y$, so there exists $U\in\beta_1$ such that $x\in U$ and $y\notin U$. Since $\leq_1$ is the specialization order of (X,τ_1), U is an $\leq_1$-upset. From $U\in\beta_1$ it follows that $U^c\in\beta_2\subseteq\tau$. So both U and U^c are open in (X,τ), and so U is clopen in (X,τ). Therefore, U is a clopen upset of $(X,\tau,\leq)$, implying that $(X,\tau,\leq)$ satisfies PSA. Thus, $(X,\tau,\leq)$ is a Priestley space.

(i) We already showed that $\beta_1\subseteq\mathsf{CpUp}(X,\tau,\leq)$. Let $A\in\mathsf{CpUp}(X,\tau,\leq)$. We show that $A=\bigcup\{U\in\beta_1 \mid U\subseteq A\}$. That $\bigcup\{U\in\beta_1\mid U\subseteq A\}\subseteq A$ is obvious. Let $x\in A$. Since A is an upset, for each $y\in A^c$ we have $x\not\leq y$. Therefore, $x\not\leq_1 y$, and as β_1 is a basis for (X,τ_1), there exists $U_y\in\beta_1$ such that $x\in U_y$ and $y\notin U_y$. It follows that $A^c\cap\bigcap\{U_y\mid y\in A^c\}=\emptyset$. Thus, $\{A^c\}\cup\{U_y\mid y\in A^c\}$ is a family of closed subsets of (X,τ) with the empty intersection, and as (X,τ) is compact, there are $U_1,\dots,U_n\in\beta_1$ with $A^c\cap U_1\cap\dots\cap U_n=\emptyset$. Therefore, $x\in U_1\cap\dots\cap U_n\subseteq A$. Since β_1 is closed under finite intersections, we obtain that there is $U\in\beta_1$ such that $x\in U\subseteq A$. Thus, $A=\bigcup\{U\in\beta_1\mid U\subseteq A\}$. Now since A is a closed subset of a compact space, A is compact, so it is a finite union of elements of β_1, thus $A\in\beta_1$.

(ii) Since every open upset is the union of clopen upsets of $(X, \tau, \leq)$ and β_1 is a basis for (X, τ_1), the result follows from (i).

(iv) and (v) are proved similarly to (i) and (ii).

(iii) Since closed upsets are intersections of clopen upsets of $(X, \tau, \leq)$, and clopen upsets are elements of β_1, closed upsets are intersections of elements of β_1. Because $\beta_1 = \{U^c \mid U \in \beta_2\}$, intersections of elements of β_1 are intersections of complements of elements of β_2, so are complements of unions of elements of β_2. As unions of elements of β_2 are elements of τ_2, we obtain that closed upsets are complements of elements of τ_2, so are elements of δ_2. Consequently, $\mathsf{ClUp}(X, \tau, \leq) = \delta_2$.

(vi) is proved similarly to (iii). ⊣

Proposition 3.5. *Let* (X, τ_1, τ_2) *and* (X', τ_1', τ_2') *be pairwise Stone spaces. If* $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ *is bi-continuous, then* $f : (X, \tau, \leq) \to (X', \tau', \leq')$ *is continuous and order-preserving.*

Proof. Since f is bi-continuous, the f inverse image of every element of $\tau_1' \cup \tau_2'$ is an element of $\tau_1 \cup \tau_2$. As $\tau_1' \cup \tau_2'$ is a subbasis for (X, τ'), it follows that $f : (X, \tau) \to (X', \tau')$ is continuous. Also, since the f inverse image of an element of τ_1' is an element of τ_1 and $\leq' = \leq_1'$, it follows that $f : (X, \leq) \to (X', \leq')$ is order-preserving. Thus, $f : (X, \tau, \leq) \to (X', \tau', \leq')$ is continuous and order-preserving. ⊣

We define the functor $\Phi : \mathbf{PStone} \to \mathbf{Pries}$ as follows. For (X, τ_1, τ_2) a pairwise Stone space, we put $\Phi(X, \tau_1, \tau_2) = (X, \tau, \leq)$, and for $f : (X, \tau, \leq) \to (X', \tau', \leq')$ a bi-continuous map, we put $\Phi(f) = f$. It follows from Propositions 3.4 and 3.5 that Φ is well-defined.

For $(X, \tau, \leq)$ a Priestley space, let $\tau_1 = \mathsf{OpUp}(X, \tau, \leq)$ and $\tau_2 = \mathsf{OpDo}(X, \tau, \leq)$. Clearly τ_1 and τ_2 are topologies on X.

Proposition 3.6. *If* $(X, \tau, \leq)$ *is a Priestley space, then* (X, τ_1, τ_2) *is a pairwise Stone space. Moreover:*

(i) $\beta_1 = \mathsf{CpUp}(X, \tau, \leq)$.
(ii) $\beta_2 = \mathsf{CpDo}(X, \tau, \leq)$.
(iii) $\leq = \leq_1 = \geq_2$.

Proof. Since (X, τ) is compact and $\tau_1 \cup \tau_2 \subseteq \tau$, it follows that (X, τ_1, τ_2) is pairwise compact. To show that (X, τ_1, τ_2) is pairwise Hausdorff, let x, y be two distinct points of X. Since $\leq$ is a partial order, we have $x \not\leq y$ or $y \not\leq x$. In either case, by PSA, one of the points has a clopen upset neighborhood U not containing the other. Clearly U^c is a clopen downset. Therefore, $U \in \tau_1$ and $U^c \in \tau_2$ separate x and y. Thus, (X, τ_1, τ_2) is pairwise Hausdorff. That (X, τ_1, τ_2) is pairwise zero-dimensional follows from (i), (ii), and the fact that open upsets are unions of clopen upsets and open downsets are unions of clopen downsets (see Lemma 3.2(3)). Consequently, (X, τ_1, τ_2) is a pairwise Stone space.

(i) For $U \subseteq X$ we have:

$$\begin{aligned} A \in \beta_1 \quad & \text{iff} \quad A \in \tau_1 \text{ and } A^c \in \tau_2 \\ & \text{iff} \quad A \in \mathsf{OpUp}(X, \tau, \leq) \text{ and } A^c \in \mathsf{OpDo}(X, \tau, \leq) \\ & \text{iff} \quad A \in \mathsf{CpUp}(X, \leq). \end{aligned}$$

Thus, $\beta_1 = \mathsf{CpUp}(X, \leq)$.

(ii) is proved similarly to (i).

(iii) For $x, y \in X$, by PSA, we have:

$$\begin{aligned} x \leq y \quad & \text{iff} \quad (\forall U \in \mathsf{OpUp}(X,\tau,\leq))(x \in U \Rightarrow y \in U) \\ & \text{iff} \quad (\forall U \in \tau_1)(x \in U \Rightarrow y \in U) \\ & \text{iff} \quad x \leq_1 y. \end{aligned}$$

Thus, $\leq = \leq_1$. That $\leq = \geq_2$ is proved similarly. ⊣

Proposition 3.7. *If $f : (X,\tau,\leq) \to (X',\tau',\leq')$ is continuous and order-preserving, then $f : (X,\tau_1,\tau_2) \to (X',\tau_1',\tau_2')$ is bi-continuous.*

Proof. Since f is continuous and order-preserving, $U \in \mathsf{OpUp}(X',\tau',\leq')$ implies $f^{-1}(U) \in \mathsf{OpUp}(X,\tau,\leq)$ and $U \in \mathsf{OpDo}(X',\tau',\leq')$ implies $f^{-1}(U) \in \mathsf{OpDo}(X,\tau,\leq)$. By the definition of the topologies, $\mathsf{OpUp}(X,\tau,\leq) = \tau_1$, $\mathsf{OpUp}(X',\tau',\leq') = \tau_1'$, $\mathsf{OpDo}(X,\tau,\leq) = \tau_2$, and $\mathsf{OpDo}(X',\tau',\leq') = \tau_2'$. Thus, $f : (X,\tau_1,\tau_2) \to (X',\tau_1',\tau_2')$ is bi-continuous. ⊣

Now we define Ψ : **Pries** $\to$ **PStone** as follows. For $(X,\tau,\leq)$ a Priestley space, we put $\Psi(X,\tau,\leq) = (X,\tau_1,\tau_2)$, and for $f : (X,\tau,\leq) \to (X',\tau',\leq')$ continuous and order-preserving, we put $\Psi(f) = f$. It follows from Propositions 3.6 and 3.7 that Ψ is well-defined.

Theorem 3.8. *The functors Φ and Ψ establish an isomorphism between the categories* **PStone** *and* **Pries**.

Proof. We already verified that Φ and Ψ are well-defined. That they are natural is easy to see. Moreover, for each pairwise Stone space (X,τ_1,τ_2), by Proposition 3.4, we have $\Psi\Phi(X,\tau_1,\tau_2) = \Psi(X,\tau,\leq) = (X,\mathsf{OpUp}(X,\tau,\leq),\mathsf{OpDo}(X,\tau,\leq)) = (X,\tau_1,\tau_2)$. Also, for each Priestley space $(X,\tau,\leq)$, by Lemma 3.2(4) and Proposition 3.6, we have $\Phi\Psi(X,\tau,\leq) = \Phi(X,\tau_1,\tau_2) = (X,\tau_1 \vee \tau_2,\leq_1) = (X,\tau,\leq)$. Thus, Φ and Ψ establish an isomorphism between **PStone** and **Priest**. ⊣

4. Pairwise Stone spaces and spectral spaces

For a topological space (X,τ), let $\mathcal{E}(X,\tau)$ denote the set of *compact open* subsets of (X,τ). We recall that (X,τ) is *coherent* if $\mathcal{E}(X,\tau)$ is closed under finite intersections and forms a basis for the topology. We also recall that a subset A of X is *irreducible* if $A = F \cup G$, with F,G closed, implies that $A = F$ or $A = G$, and that (X,τ) is *sober* if every irreducible closed subset of (X,τ) is the closure of a point. Clearly a closed subset of X is irreducible iff it is a join-prime element in the lattice of closed subsets of (X,τ). We will use this fact in the proof of Proposition 4.2.

Definition 4.1. [16, p. 43] *A topological space (X,τ) is called a* spectral space *if (X,τ) is compact, T_0, coherent, and sober.*

Let (X,τ) and (X',τ') be two spectral spaces. We recall [16, p. 43] that a map $f : (X,\tau) \to (X',\tau')$ is a *spectral map* if $U \in \mathcal{E}(X',\tau')$ implies $f^{-1}(U) \in \mathcal{E}(X,\tau)$. Clearly every spectral map is continuous.

Let **Spec** denote the category of spectral spaces and spectral maps. It follows from [6] that **Spec** is isomorphic to **Pries**. Thus, by Theorem 3.8, **Spec** is isomorphic to **PStone**. Nevertheless, we give a direct proof of this result. On the one hand, it will underline the utility of sobriety in the definition of a spectral space; on the other hand, it will provide a more natural proof of Cornish's result that **Pries** and **Spec** are isomorphic, by first establishing the intermediate isomorphisms of **Pries** and **PStone** and **PStone** and **Spec**.

Proposition 4.2. *If (X,τ_1,τ_2) is a pairwise Stone space, then (X,τ_1) is a spectral space. Moreover, $\mathcal{E}(X,\tau_1) = \beta_1$.*

Proof. Since (X, τ_1, τ_2) is pairwise compact, it is immediate that (X, τ_1) is compact. It follows from Lemma 2.5 that (X, τ_1) is T_0. We show that $\mathcal{E}(X, \tau_1) = \beta_1$. By Proposition 2.9, $\beta_1 = \tau_1 \cap \delta_2 \subseteq \tau_1 \cap \sigma_1 = \mathcal{E}(X, \tau_1)$. Conversely, suppose that $U \in \mathcal{E}(X, \tau_1)$. Since β_1 is a basis for (X, τ_1), we have U is the union of elements of β_1. As U is compact, it is a finite union of elements of β_1, thus belongs to β_1 because β_1 is closed under finite unions. Therefore, $\mathcal{E}(X, \tau_1) = \beta_1$. It follows that $\mathcal{E}(X, \tau_1)$ is closed under finite intersections and forms a basis for the topology. Therefore, (X, τ_1) is coherent. To show that (X, τ_1) is sober, let F be a join-prime element in the lattice of closed subsets of (X, τ_1). We show that F is equal to the closure in (X, τ_1) of a point of F. If not, then for each $x \in F$ there exists $y \in F$ such that $y \notin \mathrm{Cl}_1(x)$. Therefore, there exists $U_y \in \beta_1$ such that $y \in U_y$ and $x \notin U_y$. Let $U_x = U_y^c$. Then $x \in U_x \in \beta_2$, $y \notin U_x$, and F is covered by the family $\{U_x \mid x \in F\}$. Since $F \in \delta_1 \subseteq \sigma_2$, there exist $x_1, \ldots, x_n \in F$ such that $F \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. As F is join-prime in δ_1 and for each i we have $U_{x_i} \in \beta_2 \subseteq \delta_1$, there exists k such that $F \subseteq U_{x_k}$. On the other hand, the y_k corresponding to x_k belongs to F and does not belong to U_{x_k}, a contradiction. Thus, there is $x \in F$ such that $F = \mathrm{Cl}_1(x)$. Consequently, (X, τ_1) is sober, and so (X, τ_1) is a spectral space. ⊣

Proposition 4.3. *Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be two pairwise Stone spaces. If $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous, then $f : (X, \tau_1) \to (X', \tau_1')$ is spectral.*

Proof. Since f is bi-continuous, by Proposition 4.2, we have:

$$\begin{array}{rcl} U \in \mathcal{E}(X', \tau_1') & \Rightarrow & U \in \beta_1' \\ & \Rightarrow & U \in \tau_1' \cap \delta_2' \\ & \Rightarrow & f^{-1}(U) \in \tau_1 \cap \delta_2 \\ & \Rightarrow & f^{-1}(U) \in \beta_1 \\ & \Rightarrow & f^{-1}(U) \in \mathcal{E}(X, \tau_1). \end{array}$$

Thus, f is spectral. ⊣

We define the functor $\mathsf{F} : \mathbf{PStone} \to \mathbf{Spec}$ as follows. For a pairwise Stone space (X, τ_1, τ_2), we put $\mathsf{F}(X, \tau_1, \tau_2) = (X, \tau_1)$, and for $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ bi-continuous, we put $\mathsf{F}(f) = f$. It follows from Propositions 4.2 and 4.3 that F is well-defined. Note that F is a forgetful functor, forgetting the topology τ_2.

For (X, τ) a spectral space, let $\tau_1 = \tau$ and τ_2 be the topology generated by the basis $\Delta(X, \tau) = \{U^c \mid U \in \mathcal{E}(X, \tau)\}$.

Remark 4.4. Let (X, τ) be a topological space. We recall (see, e.g., [23, Def. 4.4]) that the *de Groot dual* of τ is the topology τ^* whose closed sets are generated by compact saturated sets of (X, τ). Since in a spectral space (X, τ) the compact saturated sets are exactly the intersections of compact open sets, we obtain that the topology generated by $\Delta(X, \tau)$ is exactly the de Groot dual τ^* of τ.

Proposition 4.5. *If (X, τ) is a spectral space, then (X, τ_1, τ_2) is a pairwise Stone space. Moreover:*

(i) $\beta_1 = \mathcal{E}(X, \tau)$.
(ii) $\beta_2 = \Delta(X, \tau)$.

Proof. First we show that (X, τ_1, τ_2) is pairwise compact. For this it suffices to show that any collection $K \subseteq \mathcal{E}(X, \tau) \cup \Delta(X, \tau)$ with the FIP (Finite Intersection Property) has a nonempty intersection. Let $\delta = \{F \mid F^c \in \tau\}$ denote the collection of closed subsets of

(X, τ). Since $\Delta(X, \tau) \subseteq \delta$, we have that $K \subseteq \mathcal{E}(X, \tau) \cup \delta$. To show that $\bigcap K \neq \emptyset$, by Zorn's Lemma, we extend K to a maximal subset M of $\mathcal{E}(X, \tau) \cup \delta$ with the FIP. Let C denote the intersection of all τ-closed sets in M; that is, $C = \bigcap\{F \mid F \in M \cap \delta\}$. Since (X, τ) is compact, $C \in \delta$ is nonempty. Because $\mathcal{E}(X, \tau)$ is closed under finite intersections, it is easy to see that the collection $M \cup \{C\}$ has the FIP, and as M is maximal, we have $C \in M$. We show that C is irreducible. Suppose that $C = A \cup B$ and $A, B \in \delta$. If $M \cup \{A\}$ and $M \cup \{B\}$ do not have the FIP, then there exist $A_1, \dots, A_n \in M$ with $A_1 \cap \dots \cap A_n \cap A = \emptyset$ and $B_1, \dots, B_m \in M$ with $B_1 \cap \dots \cap B_m \cap B = \emptyset$. This implies that $A_1 \cap \dots \cap A_n \cap B_1 \cap \dots \cap B_m \cap C = \emptyset$, which is a contradiction. Therefore, either $M \cup \{A\}$ or $M \cup \{B\}$ has the FIP. Since M is maximal, either $A \in M$ or $B \in M$. Because of the choice of C, this implies that either $C \subseteq A$ or $C \subseteq B$, and so either $C = A$ or $C = B$. Thus, C is irreducible. As (X, τ) is sober, $C = \mathrm{Cl}(x)$ for some $x \in X$. It is clear that x belongs to all $F \in M \cap \delta$ since $C \subseteq F$ for all such F. Moreover, for each $U \in M \cap \mathcal{E}(X, \tau)$, we have $U \cap \mathrm{Cl}(x) = U \cap C \neq \emptyset$. Since U is open in (X, τ), this implies that $x \in U$. Therefore, $x \in \bigcap M$, so $x \in \bigcap K$, as $K \subseteq M$, and so $\bigcap K \neq \emptyset$. Consequently, (X, τ_1, τ_2) is pairwise compact.

We show that $\beta_1 = \mathcal{E}(X, \tau)$ and $\beta_2 = \Delta(X, \tau)$, which establishes that (X, τ_1, τ_2) is pairwise zero-dimensional. By the definition of τ_2 we have $\mathcal{E}(X, \tau) \subseteq \delta_2$, and so $\mathcal{E}(X, \tau) \subseteq \beta_1$. Conversely, since (X, τ_1, τ_2) is pairwise compact, by Proposition 2.9, we have $\beta_1 = \tau_1 \cap \delta_2 \subseteq \tau_1 \cap \sigma_1 = \mathcal{E}(X, \tau)$. Therefore, $\beta_1 = \mathcal{E}(X, \tau)$. Moreover, $U \in \Delta(X, \tau) \iff U^c \in \mathcal{E}(X, \tau) = \beta_1 = \tau_1 \cap \delta_2 \iff U \in \delta_1 \cap \tau_2 = \beta_2$. Thus, $\beta_2 = \Delta(X, \tau)$.

Lastly, we have for granted that (X, τ_1) is T_0. Therefore, by Lemma 2.5, (X, τ_1, τ_2) is pairwise T_2, so a pairwise Stone space, which concludes the proof. ⊣

Proposition 4.6. *Let (X, τ) and (X', τ') be two spectral spaces. If $f : (X, \tau) \to (X', \tau')$ is a spectral map, then $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous.*

Proof. Since f is spectral, $f : (X, \tau_1) \to (X', \tau_1')$ is continuous. Moreover, for $U \in \beta_2'$ we have $U^c \in \beta_1'$. Therefore, $f^{-1}(U) = f^{-1}((U^c)^c) = f^{-1}(U^c)^c \in \beta_2$ since $f^{-1}(U^c) \in \beta_1$, as f is spectral. Consequently, $f : (X, \tau_2) \to (X', \tau_2')$ is continuous, and so $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous. ⊣

Now we define the functor $\mathsf{G} : \mathbf{Spec} \to \mathbf{PStone}$ as follows. For a spectral space (X, τ), we put $\mathsf{G}(X, \tau) = (X, \tau_1, \tau_2)$, and for $f : (X, \tau) \to (X', \tau')$ a spectral map, we put $\mathsf{G}(f) = f$. It follows from Propositions 4.5 and 4.6 that G is well-defined.

Theorem 4.7. *The functors F and G establish an isomorphism between the categories* **PStone** *and* **Spec**.

Proof. We already verified that F and G are well-defined. That they are natural is easy to see. Moreover, for each pairwise Stone space (X, τ_1, τ_2) we have $\mathsf{GF}(X, \tau_1, \tau_2) = \mathsf{G}(X, \tau_1) = (X, \tau_1, \tau_2)$, by Proposition 4.2. Also, for each spectral space (X, τ) we have $\mathsf{FG}(X, \tau) = \mathsf{F}(X, \tau_1, \tau_2) = (X, \tau_1) = (X, \tau)$. Thus, F and G establish an isomorphism between **PStone** and **Spec**. ⊣

Putting Theorems 3.8 and 4.7 together, we obtain that the three categories **Pries**, **PStone**, and **Spec** are isomorphic. As we pointed out in the introduction, this can be viewed as a particular case of a more general result of [14, Ch. VI-6] that the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces are isomorphic. It appears to be an interesting question to investigate how far the above isomorphisms can be pushed. In other words, what are the largest categories of ordered topological spaces, bitopological spaces, and sober spaces which are still isomorphic?

5. Distributive lattices and pairwise Stone spaces

Since **PStone** is isomorphic to **Spec** and **Spec** is dually equivalent to **DLat**, it follows that **PStone** is also dually equivalent to **DLat**. We give an explicit proof of this result. It will show that of the dual equivalences of **DLat** with **Spec**, **Pries**, and **PStone**, the dual equivalence of **DLat** with **PStone** is the easiest to establish. Indeed, as we will see below, the proof of compactness of the bitopoligical dual of a bounded distributive lattice L does not require the use of Alexander's Lemma, hence is simpler than in the Priestley case; moreover, the complicated proof of sobriety of the dual spectral space of L is completely avoided in the bitopological setting.

Let L be a bounded distributive lattice and let $X = \mathsf{pf}(L)$ be the set of prime filters of L. We define $\phi_+, \phi_- : L \to \wp(X)$ by

$$\phi_+(a) = \{x \in X \mid a \in x\} \text{ and } \phi_-(a) = \{x \in X \mid a \notin x\}.$$

If we think of L as a Lindenbaum algebra and of $a \in L$ as (an equivalence class of) a formula, then we can think of $\phi_+(a)$ as the set of points a is true at, and of $\phi_-(a)$ as the set of points a is false at. It is easy to check that $\phi_+(a) = \phi_-(a)^c$, and that the following identities hold:

$$\begin{array}{ll}
1_+ : \ \phi_+(0) = \emptyset, & 1_- : \ \phi_-(0) = X, \\
2_+ : \ \phi_+(1) = X, & 2_- : \ \phi_-(1) = \emptyset, \\
3_+ : \ \phi_+(a \wedge b) = \phi_+(a) \cap \phi_+(b), & 3_- : \ \phi_-(a \wedge b) = \phi_-(a) \cup \phi_-(b), \\
4_+ : \ \phi_+(a \vee b) = \phi_+(a) \cup \phi_+(b), & 4_- : \ \phi_-(a \vee b) = \phi_-(a) \cap \phi_-(b).
\end{array}$$

Let $\beta_+ = \phi_+[L] = \{\phi_+(a) \mid a \in L\}$, $\beta_- = \phi_-[L] = \{\phi_-(a) \mid a \in L\}$, τ_+ be the topology generated by β_+, and τ_- be the topology generated by β_-.

Proposition 5.1. *(X, τ_+, τ_-) is a pairwise Stone space.*

Proof. We start by showing that (X, τ_+, τ_-) is pairwise Hausdorff. Suppose that $x \neq y$. Without loss of generality we may assume that $x \not\subseteq y$. Therefore, there exists $a \in L$ with $a \in x$ and $a \notin y$. Thus, $x \in \phi_+(a) \in \tau_+$ and $y \in \phi_-(a) \in \tau_-$. Since $\phi_-(a) = \phi_+(a)^c$, $\phi_+(a)$ and $\phi_-(a)$ are disjoint. Consequently, (X, τ_+, τ_-) is pairwise Hausdorff.

Next we show that (X, τ_+, τ_-) is pairwise compact. For this it is sufficient to show that for each cover of X by elements of $\beta_+ \cup \beta_-$, there is a finite subcover. Suppose that $X = \bigcup\{\phi_+(a_i) \mid i \in I\} \cup \bigcup\{\phi_-(b_j) \mid j \in J\}$ for some $a_i, b_j \in L$. Let Δ be the ideal generated by $\{a_i \mid i \in I\}$ and ∇ be the filter generated by $\{b_j \mid j \in J\}$. If $\Delta \cap \nabla = \emptyset$, then by the prime filter lemma, there is a prime filter x of L such that $\nabla \subseteq x$ and $x \cap \Delta = \emptyset$. Therefore, $x \in \phi_+(b_j)$ and $x \in \phi_-(a_i)$ for each $j \in J$ and $i \in I$. Thus, $x \notin \phi_-(b_j)$ and $x \notin \phi_+(a_i)$ for each $j \in J$ and $i \in I$. Consequently, $\{\phi_+(a_i) \mid i \in I\} \cup \{\phi_-(b_j) \mid j \in J\}$ is not a cover of X, a contradiction. This shows that $\nabla \cap \Delta \neq \emptyset$, and so there exist $b_{j_1}, \ldots, b_{j_n}$ and $a_{i_1}, \ldots, a_{i_m}$ such that $b_{j_1} \wedge \cdots \wedge b_{j_n} \leq a_{i_1} \vee \cdots \vee a_{i_m}$. Therefore, $\phi_+(b_{j_1}) \cap \cdots \cap \phi_+(b_{j_n}) \subseteq \phi_+(a_{i_1}) \cup \cdots \cup \phi_+(a_{i_m})$, implying that $\phi_-(b_{j_1}) \cup \ldots \phi_-(b_{j_n}) \cup \phi_+(a_{i_1}) \cup \cdots \cup \phi_+(a_{i_m}) = X$. Thus, $\{\phi_+(a_{i_1}), \ldots, \phi_+(a_{i_m}), \phi_-(b_{j_1}), \ldots, \phi_-(b_{j_n})\}$ is a finite subcover of $\{\phi_+(a_i) \mid i \in I\} \cup \{\phi_-(b_j) \mid j \in J\}$, and so (X, τ_+, τ_-) is pairwise compact.

Let δ_+ denote the set of closed subsets and σ_+ denote the set of compact subsets of (X, τ_+); δ_- and σ_- are defined similarly. We show that $\beta_+ = \tau_+ \cap \delta_-$. If $U \in \beta_+$, then it is clear that $U \in \tau_+$. Moreover, since $U = \phi_+(a)$ for some $a \in L$, we have $U^c = \phi_-(a)$, and so $U^c \in \beta_-$. Thus, $U \in \delta_-$, so $U \in \tau_+ \cap \delta_-$, and so $\beta_+ \subseteq \tau_+ \cap \delta_-$. Conversely, let $U \in \tau_+ \cap \delta_-$. Since (X, τ_+, τ_-) is pairwise compact, by Proposition 2.9, $U \in \tau_+ \cap \sigma_+$. As β_+ is a basis for τ_+, we have that U is a union of elements of β_+. Because U is compact, it is a finite such union, thus an element of β_+ as β_+ is closed under finite unions. Consequently, $\tau_+ \cap \delta_- \subseteq \beta_+$, and so $\beta_+ = \tau_+ \cap \delta_-$. A similar argument shows that $\beta_- = \tau_- \cap \delta_+$. It follows that (X, τ_+, τ_-) is pairwise zero-dimensional, and so (X, τ_+, τ_-) is a pairwise Stone space. ⊣

For a bounded lattice homomorphism $h : L \to L'$, let $f_h : \mathsf{pf}(L') \to \mathsf{pf}(L)$ be given by $f_h(x) = h^{-1}(x)$. It is easy to check that f_h is well-defined.

Proposition 5.2. *The map f_h is bi-continuous.*

Proof. Let $a \in L$. Then it is easy to verify that $f_h^{-1}(\phi_+(a)) = {\phi_+}'(ha)$ and $f_h^{-1}(\phi_-(a)) = {\phi_-}'(ha)$. Therefore, the inverse image of each element of β_+ is in ${\beta_+}'$ and the inverse image of each element of β_- is in ${\beta_-}'$. Thus, f_h is bi-continuous. ⊣

This allows us to define the contravariant functor $(-)_* : \mathbf{DLat} \to \mathbf{PStone}$ as follows. For a bounded distributive lattice L, we let $L_* = (X, \tau_+, \tau_-)$, where $X = \mathsf{pf}(L)$, τ_+ is the topology generated by the basis $\beta_+ = \phi_+[L]$, and τ_- is the topology generated by the basis $\beta_- = \phi_-[L]$. For $h \in \hom(L, L')$, we let $h_* = h^{-1}$. It follows from Propositions 5.1 and 5.2 that the functor $(-)_*$ is well-defined.

For a pairwise Stone space (X, τ_1, τ_2) it is easy to see that $(\beta_1, \cap, \cup, \emptyset, X)$ is a bounded distributive lattice. (Note that $(\beta_2, \cap, \cup, \emptyset, X)$ is also a bounded distributive lattice dually isomorphic to $(\beta_1, \cap, \cup, \emptyset, X)$.) If $f : X \to X'$ is a bi-continuous map, then for each $U \in \beta'_1$, we have $U \in \tau'_1 \cap \delta'_2$. Since f is bi-continuous, $f^{-1}(U) \in \tau_1 \cap \delta_2$. Therefore, $f^{-1}(U) \in \beta_1$. Moreover, it is clear that $f^{-1} : \beta'_1 \to \beta_1$ is a bounded lattice homomorphism. We define the contravariant functor $(-)^* : \mathbf{PStone} \to \mathbf{DLat}$ as follows. For a pairwise Stone space (X, τ_1, τ_2), we let $(X, \tau_1, \tau_2)^* = (\beta_1, \cap, \cup, \emptyset, X)$, and for $f \in \hom(X, X')$, we let $f^* = f^{-1}$. Then the functor $(-)^*$ is well-defined.

Theorem 5.3. *The functors $(-)_*$ and $(-)^*$ establish a dual equivalence between* **DLat** *and* **PStone**.

Proof. For a bounded distributive lattice L, we have ${L_*}^* = \phi_+[L]$, and so ϕ_+ is a lattice isomorphism from L to ${L_*}^*$. For a pairwise Stone space (X, τ_1, τ_2), let $\psi : X \to {X^*}_*$ be given by $\psi(x) = \{U \in X^* \mid x \in U\}$. It is easy to see that ψ is well-defined. Since X is pairwise Hausdorff, ψ is 1-1. To see that ψ is onto, let P be a prime filter of β_1. We let $Q = \{V \in \beta_2 \mid Q^c \notin P\}$. It is easy to see that Q is a prime filter of β_2, and that $P \cup Q$ has the FIP. Since X is pairwise compact and pairwise Hausdorff, there is $x \in X$ such that $\bigcap(P \cup Q) = \{x\}$. Therefore, $\psi(x) = P$, and so ψ is onto. Moreover, for $U \in \beta_1$ we have $\psi^{-1}(\phi_+(U)) = U \in \beta_1$ and $\psi^{-1}(\phi_-(U)) = U^c \in \beta_2$. Therefore, f is bi-continuous. Furthermore, for $U \in \beta_1$, because ψ is a bijection, $\psi^{-1}(\phi_+(U)) = U$ implies $\psi(U) = \phi_+(U)$, and $\psi^{-1}(\phi_-(U)) = U^c$ implies $\psi(U^c) = \phi_-(U)$. Thus, f is bi-open, and so f is a bi-homeomorphism from X to ${X^*}_*$. That the functors $(-)_*$ and $(-)^*$ are natural is standard to prove. Consequently, $(-)_*$ and $(-)^*$ establish a dual equivalence between **DLat** and **PStone**. ⊣

Remark 5.4. It is worth pointing out that as in the case of the spectral and Priestley dualities, the dual equivalence between **DLat** and **PStone** is also induced by the *schizophrenic object* $\mathbf{2} = \{0, 1\}$. It has many lives: In **DLat** it is the two-element lattice; in **Spec** it is the *Sierpinski space* with the spectral topology $\tau_1 = \{\emptyset, \{1\}, \{0, 1\}\}$; in **Pries** it is the two-element ordered topological space with the discrete topology and the order $\leq$ given by $x \leq y$ iff $x = y$ or $x = 0$ and $y = 1$; finally in **PStone** it is the two element bitopological space with two Sierpinski topologies τ_1 and $\tau_2 = \{\emptyset, \{0\}, \{0, 1\}\}$.

6. Duality

In this section we use the isomorphism of **Pries**, **PStone**, and **Spec**, and their dual equivalence to **DLat** to obtain the dual description of the algebraic concepts important for the study of distributive lattices. In particular, we give the dual descriptions of filters, ideals, homomorphic images, sublattices, canonical completions, and MacNeille completions of bounded distributive lattices. We also give the dual description of complete distributive lattices. The dual description of these concepts by means of Priestley spaces is known. Some of these concepts have also been described by means of spectral spaces. We complete the picture by giving the spectral description of the remaining concepts as well as describing them all by means of pairwise Stone spaces. At the end of the section we give a table, which serves as a dictionary of duality theory for distributive lattices, complementing the dictionary given in [27].

6.1. **Filters and ideals.** We start by the dual description of filters, prime filters, and maximal filters, as well as ideals, prime ideals, and maximal ideals of bounded distributive lattices by means of Priestley spaces.

Let L be a bounded distributive lattice and let $(X, \tau, \le)$ be the Priestley space of L. We recall that the poset $(\mathsf{Fi}(L), \supseteq)$ of filters of L is isomorphic to the poset $(\mathsf{ClUp}(X), \subseteq)$ of closed upsets of X, that the poset $(\mathsf{Id}(L), \subseteq)$ of ideals of L is isomorphic to the poset $(\mathsf{OpUp}(X), \subseteq)$ of open upsets of X, and that the isomorphisms are obtained as follows. With each filter F of L we associate the closed upset $C_F = \bigcap\{\varphi(a) \mid a \in L\}$ of X, and with each closed upset C of X we associate the filter $F_C = \{a \in L \mid C \subseteq \varphi(a)\}$ of L. Then $F \subseteq G$ iff $C_F \supseteq C_G$, $F_{C_F} = F$, and $C_{F_C} = C$. Therefore, $(\mathsf{Fi}(L), \supseteq)$ is isomorphic to $(\mathsf{ClUp}(X), \subseteq)$. Also, with each ideal I of L we associate the open upset $U_I = \bigcup\{\varphi(a) \mid a \in I\}$ of X, and with each open upset U of X we associate the ideal $I_U = \{a \in L \mid \varphi(a) \subseteq U\}$ of L. Then $I \subseteq J$ iff $U_I \subseteq U_J$, $I_{U_I} = I$, and $U_{I_U} = U$. Thus, $(\mathsf{Id}(L), \subseteq)$ is isomorphic to $(\mathsf{OpUp}(X), \subseteq)$.

Let (X, τ_1, τ_2) be the pairwise Stone space corresponding to $(X, \tau, \le)$. By Proposition 3.6, $\beta_1 = \mathsf{CpUp}(X)$ and $\beta_2 = \mathsf{CpDo}(X)$. Therefore, $\tau_1 = \mathsf{OpUp}(X)$ and $\tau_2 = \mathsf{OpDo}(X)$, and so $\delta_1 = \mathsf{ClDo}(X)$ and $\delta_2 = \mathsf{ClUp}(X)$. Thus, $(\mathsf{Fi}(L), \supseteq)$ is isomorphic to $(\delta_2, \subseteq)$ and $(\mathsf{Id}(L), \subseteq)$ is isomorphic to $(\tau_1, \subseteq)$. Let (X, τ_1) be the spectral space corresponding to (X, τ_1, τ_2). Then clearly $(\mathsf{Id}(L), \subseteq)$ is isomorphic to the poset of τ_1-open sets. In order to characterize $(\mathsf{Fi}(L), \supseteq)$ in terms of (X, τ_1), we recall [14, Def. O-5.3] that a subset A of a topological space is *saturated* if it is an intersection of open subsets of the space; alternatively, A is saturated if it is an upset in the specialization order. We define A to be *co-saturated* if A is a union of closed subsets; alternatively, A is co-saturated if it is a downset in the specialization order.

Let $(X, \tau, \le)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. Then it is clear that for $A \subseteq X$, we have that the following four conditions are equivalent: (i) A is an upset of $(X, \tau, \le)$, (ii) A is a τ_1-saturated subset of (X, τ_1, τ_2), (iii) A is a τ_2-co-saturated subset of (X, τ_1, τ_2), and (iv) A is a saturated subset of (X, τ_1). Similarly, for $B \subseteq X$, we have that the following four conditions are equivalent: (i) B is a downset of $(X, \tau, \le)$, (ii) B is a τ_1-co-saturated subset of (X, τ_1, τ_2), (iii) B is a τ_2-saturated subset of (X, τ_1, τ_2), and (iv) B is a co-saturated subset of (X, τ_1).

For a pairwise Stone space (X, τ_1, τ_2) and for $i = 1, 2$, let $\mathsf{S}_i(X)$ denote the set of τ_i-saturated sets and $\mathsf{CS}_i(X)$ denote the set of τ_i-co-saturated sets. Then $\mathrm{Up}(X) = \mathsf{S}_1(X) =$

$\mathsf{CS}_2(X)$ and $\mathrm{Do}(X) = \mathsf{CS}_1(X) = \mathsf{S}_2(X)$. This gives us the following characterization of closed upsets and closed downsets of $(X, \tau, \leq)$.

Theorem 6.1. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. For $C \subseteq X$, the following conditions are equivalent:*

(1) *C is a closed upset of $(X, \tau, \leq)$.*
(2) *C is a τ_2-closed set of (X, τ_1, τ_2).*
(3) *C is a compact saturated set of (X, τ_1).*

Proof. As we already observed, (1)⇔(2) follows from Proposition 3.6. Next we show that (1)⇒(3). Since C is an upset of X, C is saturated in (X, τ_1). As C is closed in (X, τ) and (X, τ) is Hausdorff, C is a compact subset of (X, τ). Therefore, C is also compact in (X, τ_1). Thus, C is compact and saturated in (X, τ_1). Finally, we show that (3)⇒(1). Since C is saturated in (X, τ_1), C is an upset of X. We show that C is closed in (X, τ). Let $x \notin C$. Then for each $c \in C$ we have $c \not\leq x$. Therefore, there is a clopen upset U_c of X such that $c \in U_c$ and $x \notin U_c$. Thus, $C \subseteq \bigcup\{U_c \mid c \in C\}$. By Propositions 3.6 and 4.2, each U_c belongs to $\mathcal{E}(X, \tau_1)$. Since C is compact, there are $c_1, \dots c_n \in C$ such that $C \subseteq U_{c_1} \cup \cdots \cup U_{c_n}$. But then $V = U_{c_1}^c \cap \cdots \cap U_{c_n}^c$ is a clopen downset of X containing x and having the empty intersection with C. Thus, C is closed. ⊣

A similar argument gives us:

Theorem 6.2. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. For $D \subseteq X$, the following conditions are equivalent:*

(1) *D is a closed downset of $(X, \tau, \leq)$.*
(2) *D is a τ_1-closed set of (X, τ_1, τ_2).*
(3) *D is a compact saturated set of (X, τ_2).*

For a pairwise Stone space (X, τ_1, τ_2) and $i = 1, 2$, let $\mathsf{KS}_i(X)$ denote the set of compact saturated subsets of X. Then the following characterization of filters and ideals of a bounded distributive lattice is an immediate consequence of the results obtained above.

Corollary 6.3. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. Then:*

(1) $(\mathsf{Fi}(L), \supseteq) \simeq (\mathsf{ClUp}(X), \subseteq) = (\delta_2, \subseteq) = (\mathsf{KS}_1(X), \subseteq)$.
(2) $(\mathsf{Id}(L), \subseteq) \simeq (\mathsf{OpUp}(X), \subseteq) = (\tau_1, \subseteq)$.

Remark 6.4. Corollary 6.3(1) is a particular case of the celebrated Hofmann-Mislove theorem. To see this, let X be a sober space. We recall that a filter F of the lattice τ of open subsets of X is *Scott open* if for a family $\{U_i \mid i \in I\}$ of open subsets of X, from $\bigcup\{U_i \mid i \in I\} \in F$ it follows that there exist $i_1, \dots, i_n \in I$ such that $U_{i_1} \cup \cdots \cup U_{i_n} \in F$. Let $\mathsf{SFi}(\tau)$ denote the set of Scott open filters of τ. Then the Hofmann-Mislove theorem states that $(\mathsf{SFi}(\tau), \supseteq)$ is isomorphic to $(\mathsf{KS}(X), \subseteq)$. Observing that if X is spectral, then $(\mathsf{SFi}(\tau), \supseteq)$ is actually isomorphic to $(\mathsf{Fi}(\mathcal{E}(X)), \supseteq)$, we see that Corollary 6.3(1) expresses the Hofmann-Mislove theorem in the particular case of spectral spaces.

Now we turn to the dual description of prime filters and prime ideals of L. Let $(X, \tau, \leq)$ be the Priestley space of L. It is well-known that a filter F of L is prime iff $C_F = {\uparrow}x$ for some $x \in X$, and that an ideal I of L is prime iff $U_I = (\downarrow x)^c$ for some $x \in X$. Now we give the dual description of prime filters and prime ideals of L by means of pairwise Stone and spectral spaces of L.

Lemma 6.5. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. Then for each $A \subseteq X$ we have:*
(1) $\mathrm{Cl}_1(A) = \downarrow\mathrm{Cl}(A)$.
(2) $\mathrm{Cl}_2(A) = \uparrow\mathrm{Cl}(A)$.

Proof. (1) We have $\mathrm{Cl}_1(A) = \bigcap\{B \in \delta_1 \mid A \subseteq B\} = \bigcap\{B \in \mathsf{ClDo}(X) \mid A \subseteq B\}$. By Lemma 3.2(2), $\downarrow\mathrm{Cl}(A)$ is a closed downset, and clearly $A \subseteq \downarrow\mathrm{Cl}(A)$. Therefore, $\mathrm{Cl}_1(A) \subseteq \downarrow\mathrm{Cl}(A)$. Conversely, suppose that $x \notin \mathrm{Cl}_1(A)$. Then there is $U \in \tau_1$ such that $x \in U$ and $U \cap A = \emptyset$. Since $\tau_1 = \mathsf{OpUp}(X)$, then U is an open upset of X. As U is open in (X, τ), from $U \cap A = \emptyset$ it follows that $U \cap \mathrm{Cl}(A) = \emptyset$. Because U is an upset, $U \cap \mathrm{Cl}(A) = \emptyset$ implies $U \cap \downarrow\mathrm{Cl}(A) = \emptyset$. Thus, $x \notin \downarrow\mathrm{Cl}(A)$, and so $\mathrm{Cl}_1(A) = \downarrow\mathrm{Cl}(A)$.
(2) is proved similarly. ⊣

Let (X, τ_1, τ_2) be a bitopological space. Following [14, Def. O-5.3], for $A \subseteq X$ and $i = 1, 2$, we define the τ_i-*saturation* of A as $\mathrm{Sat}_i(A) = \bigcap\{U \in \tau_i \mid A \subseteq U\}$. Obviously $\mathrm{Sat}_1(A) = \uparrow_1 A$ and $\mathrm{Sat}_2(A) = \downarrow_2 A$. This immediately gives us the following corollary to Lemma 6.5.

Corollary 6.6. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. Then for each closed set A of (X, τ) we have:*
(1) $\downarrow A = \mathrm{Cl}_1(A) = \mathrm{Sat}_2(A)$.
(2) $\uparrow A = \mathrm{Cl}_2(A) = \mathrm{Sat}_1(A)$.

In particular, for each $x \in X$ we have:
(1) $\downarrow x = \mathrm{Cl}_1(x) = \mathrm{Sat}_2(x)$.
(2) $\uparrow x = \mathrm{Cl}_2(x) = \mathrm{Sat}_1(x)$.

Putting these results together, we obtain the following dual description of prime filters and prime ideals of L.

Corollary 6.7. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. For a filter F of L, the following conditions are equivalent:*
(1) *F is a prime filter of L.*
(2) *$C_F = \uparrow x$ for some $x \in X$.*
(3) *$C_F = \mathrm{Cl}_2(x)$ for some $x \in X$.*
(4) *$C_F = \mathrm{Sat}_1(x)$ for some $x \in X$.*

Also, for an ideal I of L, the following conditions are equivalent:
(1) *I is a prime ideal of L.*
(2) *$U_I = (\downarrow x)^c$ for some $x \in X$.*
(3) *$U_I = [\mathrm{Cl}_1(x)]^c$ for some $x \in X$.*
(4) *$U_I = [\mathrm{Sat}_2(x)]^c$ for some $x \in X$.*

Another consequence of our results is the dual description of maximal filters and maximal ideals of L. Let $(X, \tau, \leq)$ be the Priestley space of L. We let $\max X$ and $\min X$ denote the sets of maximal and minimal points of X, respectively. From the dual description of prime filters and prime ideals of L it immediately follows that a filter F of L is maximal iff $C_F = \{x\}(= \uparrow x)$ for some $x \in \max X$, and that an ideal I of L is maximal iff $U_I = \{x\}^c(= (\downarrow x)^c)$ for some $x \in \min X$. This together with the above corollary immediately give us:

Corollary 6.8. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. For a filter F of L, the following conditions are equivalent:*

(1) *F is a maximal filter of L.*
(2) *$C_F = \{x\}$ for some $x \in X$ with $\uparrow x = \{x\}$.*
(3) *$C_F = \{x\}$ for some $x \in X$ with $\mathrm{Cl}_2(x) = \{x\}$.*
(4) *$C_F = \{x\}$ for some $x \in X$ with $\mathrm{Sat}_1(x) = \{x\}$.*

Also, for an ideal I of L, the following conditions are equivalent:

(1) *I is a maximal ideal of L.*
(2) *$U_I = \{x\}^c$ for some $x \in X$ with $\downarrow x = \{x\}$.*
(3) *$U_I = \{x\}^c$ for some $x \in X$ with $\mathrm{Cl}_1(x) = \{x\}$.*
(4) *$U_I = \{x\}^c$ for some $x \in X$ with $\mathrm{Sat}_2(x) = \{x\}$.*

6.2. **Homomorphic images.** It is well-known (see, e.g., [27, Cor. 2.5]) that homomorphic images of a bounded distributive lattice L are in 1-1 correspondence with closed subsets of the Priestley space $(X, \tau, \leq)$ of L. Now we give the dual description of homomorphic images of L in terms of the pairwise Stone space and spectral space of L.

Lemma 6.9. *Let $(X, \tau, \leq)$ be a Priestley space and let (X, τ_1, τ_2) be its corresponding pairwise Stone space. For $C \subseteq X$, the following conditions are equivalent.*

(1) *C is closed in $(X, \tau, \leq)$.*
(2) *C is compact in $(X, \tau, \leq)$.*
(3) *C is pairwise compact in (X, τ_1, τ_2).*

Proof. That (1)⇔(2) is obvious since (X, τ) is compact and Hausdorff. That (2)⇒(3) is straightforward. To see that (3)⇒(2), it follows from (3) that each cover $\{U_i \mid i \in I\}$ of C, with $U_i \in \tau_1 \cup \tau_2$, has a finite subcover. Now use Alexander's Lemma. ⊣

For a topological space (X, τ) and a subset Y of X, let τ^Y denote the subspace topology on Y; that is, $\tau^Y = \{U \cap Y \mid U \in \tau\}$.

Definition 6.10. *Let (X, τ) be a spectral space. We call a subset Y of X a* spectral subset *of X if (Y, τ^Y) is a spectral space and $U \in \mathcal{E}(X, \tau)$ implies $U \cap Y \in \mathcal{E}(Y, \tau^Y)$.*

Theorem 6.11. *Let (X, τ_1, τ_2) be a pairwise Stone space and let (X, τ_1) be its corresponding spectral space. For $Y \subseteq X$, the following conditions are equivalent.*

(1) *Y is pairwise compact in (X, τ_1, τ_2).*
(2) *Y is a spectral subset of (X, τ_1).*

Proof. (1)⇒(2): Since Y is pairwise compact, by Theorem 6.9, Y is closed in the corresponding Priestley space $(X, \tau, \leq)$. Let $\leq^Y$ denote the restriction of $\leq$ to Y. Then $(Y, \tau^Y, \leq^Y)$ is a Priestley space. By Propositions 3.6 and 4.2, (Y, τ_1^Y) is a spectral space. Let $U \in \mathcal{E}(X)$. Again using Propositions 3.6 and 4.2 we obtain $U \in \mathsf{CpUp}(X, \tau, \leq)$. Therefore, $U \cap Y \in \mathsf{CpUp}(Y, \tau^Y, \leq^Y)$. Thus, $U \cap Y \in \mathcal{E}(Y, \tau_1^Y)$, and so Y is a spectral subset of (X, τ_1).

(2)⇒(1): Let Y be a spectral subset of (X, τ_1) and let $\Delta(Y, \tau_1^Y) = \{Y - U \mid U \in \mathcal{E}(Y, \tau^Y)\}$. We show that τ_2^Y is the topology generated by $\Delta(Y, \tau_1^Y)$. For this we show that $\mathcal{E}(Y, \tau_1^Y) =$

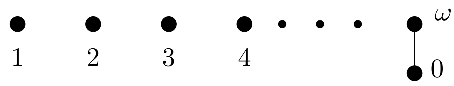

FIGURE 1

$\{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$. Since Y is a spectral subset, we have $\{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\} \subseteq \mathcal{E}(Y, \tau_1^Y)$. Conversely, suppose that $U \in \mathcal{E}(Y, \tau_1^Y)$. Then there is $V \in \tau_1$ such that $U = V \cap Y$. From $V \in \tau_1$ it follows that $V = \bigcup\{V_i \mid i \in I\}$ for some family $\{V_i \mid i \in I\} \subseteq \mathcal{E}(X, \tau_1)$. Then $U = \bigcup\{V_i \mid i \in I\} \cap Y = \bigcup\{V_i \cap Y \mid i \in I\}$. Since U is compact and $V_i \cap Y$ are open in (Y, τ_1^Y), there exist $i_1, \dots, i_n \in I$ such that $U = (V_{i_1} \cap Y) \cup \dots \cup (V_{i_n} \cap Y) = (V_{i_1} \cup \dots \cup V_{i_n}) \cap Y$. Let $W = V_{i_1} \cup \dots \cup V_{i_n}$. Since $\mathcal{E}(X, \tau_1)$ is closed under finite unions, $W \in \mathcal{E}(X, \tau_1)$. Therefore, $U = W \cap Y$ for some $W \in \mathcal{E}(X, \tau_1)$. Thus, $\mathcal{E}(Y, \tau_1^Y) \subseteq \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$, and so $\mathcal{E}(Y, \tau_1^Y) = \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$. Consequently, $\Delta(Y, \tau_1^Y) = \{Y - U \mid U \in \mathcal{E}(Y, \tau_1^Y)\} = \{Y - (V \cap Y) \mid V \in \mathcal{E}(X, \tau_1)\} = \{Y - V \mid V \in \mathcal{E}(X, \tau_1)\}$, and so τ_2^Y is the topology generated by $\Delta(Y, \tau_1^Y)$. Now, since (Y, τ_1^Y) is a spectral space, by Proposition 4.5, (Y, τ_1^Y, τ_2^Y) is pairwise compact. It follows that Y is pairwise compact in (X, τ_1, τ_2). ⊣

Now putting the above results together, we obtain the following dual description of homomorphic images of L by means of all three dual spaces of L.

Corollary 6.12. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. Then there is a 1-1 correspondence between (i) homomorphic images of L, (ii) closed subsets of $(X, \tau, \leq)$, (iii) pairwise compact subsets of (X, τ_1, τ_2), and (iv) spectral subsets of (X, τ_1).*

Proof. As follows from [27, Cor. 2.5], homomorphic images of L are in 1-1 correspondence with closed subsets of $(X, \tau, \leq)$. Lemma 6.9 and Theorem 6.11 imply that closed subsets of $(X, \tau, \leq)$ are in 1-1 correspondence with pairwise compact subsets of (X, τ_1, τ_2), which are in 1-1 correspondence with spectral subsets of (X, τ_1). The result follows. ⊣

We conclude this subsection by giving an example of a subset Y of a spectral space (X, τ) such that (Y, τ^Y) is a spectral space, but there exists $U \in \mathcal{E}(X, \tau)$ such that $U \cap Y \notin \mathcal{E}(Y, \tau^Y)$. Therefore, the condition "$U \in \mathcal{E}(X, \tau)$ implies $U \cap Y \in \mathcal{E}(Y, \tau^Y)$" can not be omitted from Definition 6.10.

Example 6.13. Let (X, τ) be the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the interval topology. Then each $n \in \omega$ is an isolated point of X and ω is the only limit point of X. For $x, y \in X$ we set $x \leq y$ iff $x = y$ or $x = 0$ and $y = \omega$ (see Figure 1). It is easy to verify that $(X, \tau, \leq)$ is a Priestley space. Let (X, τ_1, τ_2) be the corresponding pairwise Stone space and (X, τ_1) be the corresponding spectral space. We let $Y = X - \{\omega\}$. Then (Y, τ_1^Y) is a spectral space. On the other hand, $U = X - \{0\}$ is compact open in (X, τ_1), however $U \cap Y = \omega - \{0\}$ is not compact in (Y, τ^Y). Therefore, Y is not a spectral subset of (X, τ_1).

6.3. **Sublattices.** The dual description of bounded sublattices of a bounded distributive lattice by means of its Priestley space can be found in [2, 5, 31]. We will rephrase it in our terminology. We recall that a *quasi-order* Q on a set X is a reflexive and transitive relation on X. We call the pair (X, Q) a *quasi-ordered set*. For a quasi-ordered set (X, Q), we call $A \subseteq X$ a *Q-upset* of X if $x \in A$ and xQy imply $y \in A$.

Definition 6.14. *Let X be a topological space and Q be a quasi-order on X. We call Q a* Priestley quasi-order *on X if for each $x, y \in X$ with $x \not Q y$ there exists a clopen Q-upset A of X such that $x \in A$ and $y \notin A$.*

Theorem 6.15. [31, Thm. 3.7] *Let L be a bounded distributive lattice and $(X, \tau, \leq)$ be the Priestley space of L. Then there is a dual isomorphism between the poset $(\mathsf{S}_L, \subseteq)$ of bounded sublattices of L and the poset $(\mathsf{Q}_X, \subseteq)$ of Priestley quasi-orders on X extending $\leq$.*

Proof. (Sketch) For $S \in \mathsf{S}_L$, we define Q_S on X by xQ_Sy iff $x \cap S \subseteq y \cap S$. Then $Q_S \in \mathsf{Q}_X$, and $S \subseteq K$ implies $Q_K \subseteq Q_S$ for each $S, K \in \mathsf{S}_L$. Therefore, $S \mapsto Q_S$ is an order-reversing map from S_L to Q_X. For $Q \in \mathsf{Q}_X$, we let $S_Q = \{a \in L \mid \phi(a)$ is a Q-upset of $X\}$. Then S_Q is a bounded sublattice of L, and $Q \subseteq R$ implies $S_R \subseteq S_Q$ for each $Q, R \in \mathsf{Q}_X$. Thus, $Q \mapsto S_Q$ is an order-reversing map from Q_X to S_L. Moreover, $S_{Q_S} = S$ and $Q_{S_Q} = Q$ for each $S \in \mathsf{S}_L$ and $Q \in \mathsf{Q}_X$. It follows that the order-reversing maps $S \mapsto Q_S$ and $Q \mapsto S_Q$ are inverses of each other. Consequently, $(\mathsf{S}_L, \subseteq)$ is dually isomorphic to $(\mathsf{Q}_X, \subseteq)$. ⊣

Now we characterize Priestley quasi-orders extending $\leq$ by means of pairwise Stone spaces and spectral spaces.

Definition 6.16. *Let (τ_1, τ_2) and (τ_1', τ_2') be two bitopologies on X. We say that (τ_1, τ_2) is* finer *than (τ_1', τ_2') and that (τ_1', τ_2') is* coarser *than (τ_1, τ_2) if $\tau_1' \subseteq \tau_1$ and $\tau_2' \subseteq \tau_2$.*

Lemma 6.17. *Let $(X, \tau, \leq)$ be a Priestley space and (X, τ_1, τ_2) be the corresponding pairwise Stone space. Then the poset $(\mathsf{Q}_X, \subseteq)$ of Priestley quasi-orders on X is dually isomorphic to the poset $(\mathsf{Z}_X, \subseteq)$ of pairwise zero-dimensional bi-topologies on X coarser than (τ_1, τ_2).*

Proof. For a Priestley quasi-order Q on X, let τ_1^Q be the set of open Q-upsets and τ_2^Q be the set of open Q-downsets of X. Clearly (τ_1^Q, τ_2^Q) is a bitopology on X coarser than (τ_1, τ_2). Moreover, $\beta_1^Q = \tau_1^Q \cap \delta_2^Q$ is exactly the set of clopen Q-upsets of X and $\beta_2^Q = \tau_2^Q \cap \delta_1^Q$ is exactly the set of clopen Q-downsets of X. Since Q is a Priestley quasi-order, clopen Q-upsets are a basis for open Q-upsets and clopen Q-downsets are a basis for open Q-downsets. Therefore, (τ_1^Q, τ_2^Q) is pairwise zero-dimensional. For two Priestley quasi-orders Q and R on X, we show $Q \subseteq R$ implies $\tau_1^R \subseteq \tau_1^Q$ and $\tau_2^R \subseteq \tau_2^Q$. Let $U \in \tau_1^R$. Then U is an open R-upset of X. Since $Q \subseteq R$, U is also a Q-upset of X. Thus, $U \in \tau_1^Q$. That $\tau_2^R \subseteq \tau_2^Q$ is proved similarly. It follows that $Q \mapsto (\tau_1^Q, \tau_2^Q)$ is an order-reversing map from Q_X to Z_X.

Let (τ_1', τ_2') be a pairwise zero-dimensional bitopology on X coarser than (τ_1, τ_2). We define $Q_{(\tau_1', \tau_2')}$ to be the specialization order of τ_1'. Since (τ_1', τ_2') is pairwise zero-dimensional, $Q_{(\tau_1', \tau_2')}$ is the dual of the specialization order of τ_2'. Because $Q_{(\tau_1', \tau_2')}$ is a specialization order, it is clear that $Q_{(\tau_1', \tau_2')}$ is a quasi-order. From $\tau_1' \subseteq \tau_1$ it follows that $Q_{(\tau_1', \tau_2')}$ extends the specialization order of τ_1. Consequently, $Q_{(\tau_1', \tau_2')}$ extends $\leq$. We show that $Q_{(\tau_1', \tau_2')}$ is a Priestley quasi-order. If $x \not Q_{(\tau_1', \tau_2')} y$, then there exists $U \in \tau_1'$ such that $x \in U$ and $y \notin U$. Since (τ_1', τ_2') is pairwise zero-dimensional, we may assume that $U \in \beta_1'$. Therefore, U is clopen in τ. Clearly each $U \in \tau_1'$ is a $Q_{(\tau_1', \tau_2')}$-upset. Thus, there exists a clopen $Q_{(\tau_1', \tau_2')}$-upset U of X such that $x \in U$ and $y \notin U$. For $(\tau_1', \tau_2'), (\tau_1'', \tau_2'') \in \mathsf{Z}_X$, we show $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$ implies $Q_{(\tau_1'', \tau_2'')} \subseteq Q_{(\tau_1', \tau_2')}$. Let $xQ_{(\tau_1'', \tau_2'')}y$. Then $x \in U$ implies $y \in U$ for each $U \in \tau_1''$. Therefore, $x \in U$ implies $y \in U$ for each $U \in \tau_1'$. Thus, $xQ_{(\tau_1', \tau_2')}y$. It follows that $(\tau_1', \tau_2') \mapsto Q_{(\tau_1', \tau_2')}$ is an order-reversing map from Z_X to Q_X.

We show that $Q_{(\tau_1^Q, \tau_2^Q)} = Q$ and $(\tau_1^{Q_{(\tau_1', \tau_2')}}, \tau_2^{Q_{(\tau_1', \tau_2')}}) = (\tau_1', \tau_2')$ for each $Q \in \mathsf{Q}_X$ and $(\tau_1', \tau_2') \in \mathsf{Z}_X$. Indeed, $xQ_{(\tau_1^Q, \tau_2^Q)}y$ iff $(\forall U \in \tau_1^Q)(x \in U \Rightarrow y \in U)$, which is equivalent to xQy since

Q is a Priestley quasi-order. Thus, $Q_{(\tau_1^Q,\tau_2^Q)} = Q$. Moreover, $U \in \tau_1^{Q_{(\tau_1',\tau_2')}}$ iff U is an open $Q_{(\tau_1',\tau_2')}$-upset of X. Clearly $U \in \tau_1'$ implies U is an open $Q_{(\tau_1',\tau_2')}$-upset of X. Conversely, let U be an open $Q_{(\tau_1',\tau_2')}$-upset of X. We show that $U = \bigcup\{V \in \tau_1' \mid V \subseteq U\}$. Clearly $\bigcup\{V \in \tau_1' \mid V \subseteq U\} \subseteq U$. Let $x \in U$. Since U is a $Q_{(\tau_1',\tau_2')}$-upset, for each $y \in U^c$ we have $x \not\!\!Q_{(\tau_1',\tau_2')} y$. Therefore, there exists $V_y \in \tau_1'$ such that $x \in V_y$ and $y \notin V_y$. Since β_1' is a basis for τ_1', we may assume that $V_y \in \beta_1'$. Thus, $\bigcap\{V_y \mid y \in U^c\} \cap U^c = \emptyset$. Since U^c and each V_y is closed in τ and τ is compact, there exist $V_1, \dots, V_n \in \beta_1'$ such that $V_1 \cap \cdots \cap V_n \cap U^c = \emptyset$. So $x \in V_1 \cap \cdots \cap V_n \subseteq U^c$, and so $U \subseteq \bigcup\{V \in \tau_1' \mid V \subseteq U\}$. Consequently, $U \in \tau_1'$. This implies that $\tau_1^{Q_{(\tau_1',\tau_2')}} = \tau_1'$. A similar argument shows that $\tau_2^{Q_{(\tau_1',\tau_2')}} = \tau_2'$. Thus, $(\tau_1^{Q_{(\tau_1',\tau_2')}}, \tau_2^{Q_{(\tau_1',\tau_2')}}) = (\tau_1', \tau_2')$. It follows that the order-reversing maps $Q \mapsto (\tau_1^Q, \tau_2^Q)$ and $(\tau_1', \tau_2') \mapsto Q_{(\tau_1',\tau_2')}$ are inverses of each other. Thus, $(\mathsf{Q}_X, \subseteq)$ is dually isomorphic to $(\mathsf{Z}_X, \subseteq)$. ⊣

Definition 6.18. *Let τ be a spectral topology on X and let τ' be a coherent topology on X coarser than τ. We call τ'* strongly coherent *if the set $\mathcal{E}(X, \tau')$ of compact open subsets of (X, τ') is equal to the set $\tau' \cap \sigma$ of open subsets of (X, τ') that are compact in (X, τ).*

Lemma 6.19. *Let (X, τ_1, τ_2) be a pairwise Stone space and (X, τ_1) be the corresponding spectral space. Then the poset $(\mathsf{Z}_X, \subseteq)$ of pairwise zero-dimensional bitopologies (τ_1', τ_2') on X coarser than (τ_1, τ_2) is isomorphic to the poset $(\mathsf{SC}_X, \subseteq)$ of strongly coherent topologies τ_1' on X coarser than τ_1.*

Proof. Let (τ_1', τ_2') be a pairwise zero-dimensional bitopology on X coarser than (τ_1, τ_2). Then τ_1' is a topology on X coarser than τ_1. Let $\beta_1' = \tau_1' \cap \delta_2'$. We show that $\mathcal{E}(X, \tau_1') = \beta_1' = \tau_1' \cap \sigma_1$. Let $U \in \mathcal{E}(X, \tau_1')$. Since β_1' is a basis for τ_1', U is the union of elements of β_1' contained in U. As U is compact in (X, τ_1'), U is a finite union of elements of β_1', so U is an element of β_1', and so $\mathcal{E}(X, \tau_1') \subseteq \beta_1'$. Now let $U \in \beta_1'$. Because (X, τ_1, τ_2) is pairwise compact, $\delta_2 \subseteq \sigma_1$. Therefore, $\delta_2' \subseteq \delta_2 \subseteq \sigma_1$, and so $\beta_1' \subseteq \tau_1' \cap \delta_2' \subseteq \tau_1' \cap \sigma_1$. Finally, let $U \in \tau_1' \cap \sigma_1$. Since $U \in \tau_1'$ and $\mathcal{E}(X, \tau_1')$ is a basis for τ_1', U is the union of elements of $\mathcal{E}(X, \tau_1')$ contained in U. Because $U \in \sigma_1$ and $\tau_1' \subseteq \tau_1$, U is a finite union of elements of $\mathcal{E}(X, \tau_1')$. Therefore, $U \in \mathcal{E}(X, \tau_1')$, and so $\tau_1' \cap \sigma_1 \subseteq \mathcal{E}(X, \tau_1')$. Thus, $\mathcal{E}(X, \tau_1') = \beta_1' = \tau_1' \cap \sigma_1$, implying that τ_1' is a strongly coherent topology. For $(\tau_1', \tau_2'), (\tau_1'', \tau_2'') \in \mathsf{Z}_X$, if $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$, then it is obvious that $\tau_1' \subseteq \tau_1''$. It follows that $(\tau_1', \tau_2') \mapsto \tau_1'$ is an order-preserving map from Z_X to SC_X.

For a strongly coherent topology τ_1' on X coarser than τ_1, we let τ_2' be the topology generated by the basis $\Delta(X, \tau_1') = \{U^c \mid U \in \mathcal{E}(X, \tau_1')\}$. Let δ_1' denote the set of closed subsets of (X, τ_1') and δ_2' denote the set of closed subsets of (X, τ_2'). We set $\beta_1' = \tau_1' \cap \delta_2'$ and $\beta_2' = \tau_2' \cap \delta_1'$. We show that $\beta_1' = \mathcal{E}(X, \tau_1')$ and $\beta_2' = \Delta(X, \tau_1')$. It follows from the definition that $\mathcal{E}(X, \tau_1') \subseteq \beta_1'$. Conversely, $\beta_1' = \tau_1' \cap \delta_2' \subseteq \tau_1' \cap \delta_2 \subseteq \tau_1' \cap \sigma_1 = \mathcal{E}(X, \tau_1')$. Therefore, $\beta_1' = \mathcal{E}(X, \tau_1')$. Also, $U \in \Delta(X, \tau_1')$ iff $U^c \in \mathcal{E}(X, \tau_1')$ iff $U^c \in \beta_1'$ iff $U^c \in \tau_1' \cap \delta_2'$ iff $U \in \delta_1' \cap \tau_2'$ iff $U \in \beta_2'$. Thus, $\beta_2' = \Delta(X, \tau_1')$. Consequently, β_1' is a basis for τ_1' and β_2' is a basis for τ_2', and so (τ_1', τ_2') is pairwise zero-dimensional. For $\tau_1', \tau_1'' \in \mathsf{SC}_X$, we show $\tau_1' \subseteq \tau_1''$ implies $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$. Let $U \in \Delta(X, \tau_1')$. Then $U^c \in \mathcal{E}(X, \tau_1')$. Therefore, $U^c \in \tau_1' \cap \sigma_1 \subseteq \tau_1'' \cap \sigma_1$, and so $U^c \in \mathcal{E}(X, \tau_1'')$. Thus, $U \in \Delta(X, \tau_1'')$, so $\Delta(X, \tau_1') \subseteq \Delta(X, \tau_1'')$, and so $\tau_2' \subseteq \tau_2''$. It follows that $\tau_1' \mapsto (\tau_1', \tau_2')$ is an order-preserving map from SC_X to Z_X.

Finally, if $(\tau_1', \tau_2') \in \mathsf{Z}_X$, then $\mathcal{E}(X, \tau_1') = \beta_1'$, so $\Delta(X, \tau_1') = \beta_2'$, and so the composition $\mathsf{Z}_X \to \mathsf{SC}_X \to \mathsf{Z}_X$ is an identity. Moreover, it is clear that the composition $\mathsf{SC}_X \to \mathsf{Z}_X \to \mathsf{SC}_X$ is also an identity. Thus, $(\mathsf{Z}_X, \subseteq)$ is isomorphic to $(\mathsf{SC}_X, \subseteq)$. ⊣

Putting Theorem 6.15 and Lemmas 6.17 and 6.19 together, we obtain the following dual description of bounded sublattices of L by means of all three dual spaces of L.

Corollary 6.20. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then the poset $(\mathsf{S}_L, \subseteq)$ of bounded sublattices of L is dually isomorphic to the poset $(\mathsf{Q}_X, \subseteq)$ of Priestley quasi-orders on X extending $\leq$, and is isomorphic to the poset $(\mathsf{Z}_X, \subseteq)$ of pairwise zero-dimensional bitopologies on X coarser than (τ_1, τ_2), and to the poset $(\mathsf{SC}_X, \subseteq)$ of strongly coherent topologies on X coarser than τ_1.*

6.4. **Canonical completions, MacNeille completions, and complete lattices.** In the theory of completions of lattices, or more generally of posets, the MacNeille and canonical completions play a prominent role. Let L be a lattice. We recall that a subset S of L is *join-dense* in L if for each $a \in L$ we have $a = \bigvee(\downarrow a \cap S)$, and that S is *meet-dense* in L if for each $a \in L$ we have $a = \bigwedge(\uparrow a \cap S)$. We further recall that the *MacNeille completion* of L is a unique up to isomorphism complete lattice $\overline{L}$ together with a lattice embedding $i : L \to \overline{L}$ such that $i[L]$ is both join-dense and meet-dense in L. Furthermore, we recall that the *canonical completion* of L is a unique up to isomorphism complete lattice L^σ together with a lattice embedding $j : L \to L^\sigma$ such that (i) for each filter F and ideal I of L, from $F \cap I = \emptyset$ it follows that $\bigwedge j[F] \not\leq \bigvee j[I]$, (ii) the set $K_L = \{\bigwedge j[S] \mid S \subseteq L\}$ of closed elements of L^σ is join-dense in L^σ, and (iii) the set $O_L = \{\bigvee j[S] \mid S \subseteq L\}$ of open elements of L^σ is meet-dense in L^σ.

For a Priestley space $(X, \tau, \leq)$, following [15, Sec. 3], we define two maps $\mathbf{D} : \mathsf{OpUp}(X) \to \mathsf{ClUp}(X)$ and $\mathbf{J} : \mathsf{ClUp}(X) \to \mathsf{OpUp}(X)$ by $\mathbf{D}(U) = \uparrow \mathrm{Cl}(U)$ and $\mathbf{J}(K) = (\downarrow(\mathrm{Int} K)^c)^c$ for $U \in \mathsf{OpUp}(X)$ and $K \in \mathsf{ClUp}(X)$. Then it follows from [15, Lemma 3.4] that $\mathbf{D}$ and $\mathbf{J}$ form a Galois connection between $(\mathsf{OpUp}(X), \subseteq)$ and $(\mathsf{ClUp}(X), \supseteq)$. Let $\mathsf{RgOpUp}(X)$ denote the set of fixpoints of $\mathbf{J} \circ \mathbf{D}$; that is, $\mathsf{RgOpUp}(X) = \{U \in \mathsf{OpUp}(X) \mid \mathbf{JD}U = U\}$. The next theorem is well-known. The first half of it can be found in [15, Thm. 3.5], and the second half in [13, Sec. 2].

Theorem 6.21. *Let L be a bounded distributive lattice and $(X, \tau, \leq)$ be the Priestley space of L. Then $\overline{L}$ is isomorphic to $\mathsf{RgOpUp}(X)$ and L^σ is isomorphic to $\mathsf{Up}(X)$.*

Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Since $\mathsf{Up}(X) = \mathsf{S}_1(X) = \mathsf{CS}_2(X)$, we immediately obtain the following dual description of the canonical completion of L.

Theorem 6.22. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then L^σ is isomorphic to $\mathsf{Up}(X) = \mathsf{S}_1(X) = \mathsf{CS}_2(X)$.*

Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, and (X, τ_1, τ_2) be the pairwise Stone space of L. Since $\mathsf{OpUp}(X) = \tau_1$, $\mathsf{ClUp}(X) = \delta_2$, $\mathbf{D}(U) = \mathrm{Cl}_2(U)$, and $\mathbf{J}(U) = \mathrm{Int}_1(U)$ for $U \subseteq X$, we obtain that $\mathrm{Cl}_2 : \tau_1 \to \delta_2$ and $\mathrm{Int}_1 : \delta_2 \to \tau_1$ form a Galois connection between $(\tau_1, \subseteq)$ and $(\delta_2, \supseteq)$, and so the MacNeille completion $\overline{L}$ of L is isomorphic to the fixpoints of $\mathrm{Int}_1 \circ \mathrm{Cl}_2$, we denote by $\mathsf{RgOp}_{12}(X)$.

Let (X, τ_1) be the spectral space corresponding to the pairwise Stone space (X, τ_1, τ_2). Then $\delta_2 = \mathsf{KS}_1(X)$ and $\mathrm{Cl}_2(U) = \mathrm{Sat}_1\mathrm{Cl}(U)$ for $U \subseteq X$. Let $\mathrm{S}_1 = \mathrm{Sat}_1 \circ \mathrm{Cl}$. Then $\mathrm{S}_1 : \tau_1 \to \mathsf{KS}_1(X)$ and $\mathrm{Int}_1 : \mathsf{KS}_1(X) \to \tau_1$ form a Galois connection between $(\tau_1, \subseteq)$ and

$(\mathsf{KS}_1(X), \supseteq)$, and so the MacNeille completion $\overline{L}$ of L is isomorphic to the fixpoints of $\mathrm{Int}_1 \circ \mathrm{S}_1$, we denote by $\mathsf{SatOp}_1(X)$. Consequently, we obtain the following dual description of the MacNeille completion of L.

Theorem 6.23. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then $\overline{L}$ is isomorphic to $\mathsf{RgOpUp}(X) = \mathsf{RgOp}_{12}(X) = \mathsf{SatOp}_1(X)$.*

The bitopological description of $\overline{L}$ provides a nice generalization of the characterization of the MacNeille completion of a Boolean algebra B by means of the regular open subsets of the Stone space (X, τ) of B. We recall that the regular open subsets of (X, τ) are exactly the fixpoints of the composition of the maps $\mathrm{Cl} : \tau \to \delta$ and $\mathrm{Int} : \delta \to \tau$. When working with a pairwise Stone space (X, τ_1, τ_2), we consider the fixpoints of the composition of the maps Cl_2 and Int_1 between τ_1 and δ_2, respectively. Therefore, whenever $\tau_1 = \tau_2$, the pairwise Stone space (X, τ_1, τ_2) becomes the Stone space (X, τ), where $\tau = \tau_1 = \tau_2$. So $\tau_1 = \tau$, $\delta_2 = \delta$, $\mathrm{Cl}_2 = \mathrm{Cl}$, $\mathrm{Int}_1 = \mathrm{Int}$, and the fixpoints of $\mathrm{Int}_1 \circ \mathrm{Cl}_2$ are exactly the regular open subsets of (X, τ). As a corollary, we obtain the well-known dual description of the MacNeille completion of a Boolean algebra:

Corollary 6.24. *Let B be a Boolean algebra and X be the Stone space of B. Then the MacNeille completion $\overline{B}$ of B is isomorphic to the regular open subsets $\mathsf{RgOp}(X)$ of X.*

Since L is a complete lattice iff L is isomorphic to $\overline{L}$, it follows from the construction of $\overline{L}$ that L is complete iff in the dual Priestley space $(X, \tau, \leq)$ of L we have $\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$ (see [26, Prop. 16] and [15, p. 948]). Such Priestley spaces were called *extremally order disconnected* in [26, p. 521]. This together with Theorem 6.23 immediately give us the following dual description of complete distributive lattices.

Theorem 6.25. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then the following conditions are equivalent:*

(1) *L is complete.*
(2) $\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$.
(3) $\mathsf{RgOp}_{12}(X) = \beta_1$.
(4) $\mathsf{SatOp}_1(X) = \mathcal{E}(X, \tau_1)$.

In Table 1 we gather together the dual descriptions of different algebraic concepts for bounded distributive lattices by means of their Priestley spaces, pairwise Stone spaces, and spectral spaces obtained in this section. This can be thought of as a dictionary of duality theory for bounded distributive lattices, complementing the dictionary given in [27].

7. Duality for Heyting algebras

A rather natural subclass of the class of distributive lattices is the class of Heyting algebras, which plays an important role in the study of superintuitionistic logics. The first duality for Heyting algebras was developed by Esakia [8]. It is a restricted version of Priestley's duality. In this section we develop duality for Heyting algebras by means of pairwise Stone spaces and spectral spaces, thus providing the bitopological and spectral alternatives to the Esakia duality.

DLat	**Pries**	**PStone**	**Spec**
filter	closed upset	τ_2-closed set	compact saturated set
ideal	open upset	τ_1-open set	open set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	closed subset	pairwise compact subset	spectral subset
subalgebra	$Q \in \mathsf{Q}_X$	$(\tau_1', \tau_2') \in \mathsf{Z}_X$	$\tau' \in \mathsf{SC}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille completion	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

TABLE 1. Dictionary for **DLat**, **Pries**, **PStone**, and **Spec**.

We recall that a *Heyting algebra* is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ with a binary operation $\to: A^2 \to A$ such that for all $a, b, c \in A$ we have:

$$c \leq a \to b \text{ iff } a \wedge c \leq b.$$

Let **Heyt** denote the category of Heyting algebras and Heyting algebra homomorphisms. For a topological space (X, τ), let $\mathsf{Cp}(X)$ denote the set of clopen subsets of X.

Definition 7.1. *Let $(X, \tau, \leq)$ be a Priestley space. We call $(X, \tau, \leq)$ an* Esakia space *if $A \in \mathsf{Cp}(X)$ implies $\downarrow A \in \mathsf{Cp}(X)$.*

Let $(X, \leq)$ and $(X', \leq')$ be two posets. We recall that a map $f : X \to X'$ is a *p-morphism* if it is order-preserving and for each $x \in X$ and $x' \in X'$, from $f(x) \leq x'$ it follows that there is $y \in X$ such that $x \leq y$ and $f(y) = x'$. For two Esakia spaces $(X, \tau, \leq)$ and $(X', \tau', \leq')$, we call a map $f : X \to X'$ an *Esakia morphism* if it is a continuous p-morphism. Let **Esa** denote the category of Esakia spaces and Esakia morphisms. Then we have the following theorem established in [8]:

Theorem 7.2. Heyt *is dually equivalent to* **Esa**.

In fact, the same functors establishing the dual equivalence of **DLat** and **Pries** restricted to **Heyt** and **Esa**, respectively, establish the required dual equivalence. In order to describe the pairwise Stone spaces and spectral spaces dual to Heyting algebras, it is sufficient to characterize those pairwise Stone spaces and spectral spaces that correspond to Esakia spaces. As an immediate consequence of Lemma 6.9 and Theorem 6.11, we obtain:

Lemma 7.3. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. For $Y \subseteq X$, the following conditions are equivalent:*

(1) *Y is clopen in $(X, \tau, \leq)$.*
(2) *Y and Y^c are pairwise compact in (X, τ_1, τ_2).*
(3) *Y and Y^c are spectral subsets of (X, τ_1).*

Let (X, τ_1, τ_2) be a pairwise Stone space. We call $Y \subseteq X$ *pairwise clopen* if both Y and Y^c are pairwise compact in (X, τ_1, τ_2). Let $\mathsf{PC}(X)$ denote the set of pairwise clopen subsets of (X, τ_1, τ_2).

Definition 7.4. *Let (X, τ_1, τ_2) be a pairwise Stone space. We call (X, τ_1, τ_2) a* bitopological Esakia space *if $A \in \mathsf{PC}(X)$ implies $\mathrm{Cl}_1(A) \in \mathsf{PC}(X)$.*

For a pairwise Stone space (X, τ_1, τ_2), we recall that δ_1 denotes the collection of closed subsets of (X, τ_1), that δ_2 denotes the collection of closed subsets of (X, τ_2), that $\beta_1 = \tau_1 \cap \delta_2$, and that $\beta_2 = \tau_2 \cap \delta_1$.

Theorem 7.5. *Let (X, τ_1, τ_2) be a pairwise Stone space. Then (X, τ_1, τ_2) is a bitopological Esakia space iff for each $A \in \beta_1$ and $B \in \beta_2$ we have $\mathrm{Cl}_1(A \cap B) \in \beta_2$.*

Proof. Let $(X, \tau, \leq)$ be the Priestley space corresponding to (X, τ_1, τ_2). Suppose that (X, τ_1, τ_2) is a bitopological Esakia space, $A \in \beta_1$, and $B \in \beta_2$. Then $A \in \delta_2$ and $A^c \in \delta_1$. Therefore, both A and A^c are closed in $(X, \tau, \leq)$. A similar argument shows that both B and B^c are closed in $(X, \tau, \leq)$. Thus, both $A \cap B$ and $(A \cap B)^c = A^c \cup B^c$ are closed in $(X, \tau, \leq)$. By Lemma 6.9, both $A \cap B$ and $(A \cap B)^c$ are pairwise compact in $(X, \tau, \leq)$, implying that $A \cap B \in \mathsf{PC}(X)$. Since (X, τ_1, τ_2) is a bitopological Esakia space, we have $\mathrm{Cl}_1(A \cap B) \in \mathsf{PC}(X)$. By Lemma 7.3, $\mathrm{Cl}_1(A \cap B)$ is clopen in $(X, \tau, \leq)$. Moreover, since $\leq$ is the specialization order of (X, τ_1), we have that $\mathrm{Cl}_1(A \cap B)$ is a downset of $(X, \tau, \leq)$. Therefore, $\mathrm{Cl}_1(A \cap B) \in \mathsf{CpDo}(X)$. By Proposition 3.4, $\mathsf{CpDo}(X) = \beta_2$. Thus, $\mathrm{Cl}_1(A \cap B) \in \beta_2$.

Conversely, suppose that (X, τ_1, τ_2) is a pairwise Stone space and for each $A \in \beta_1$ and $B \in \beta_2$ we have $\mathrm{Cl}_1(A \cap B) \in \beta_2$. Let $A \in \mathsf{PC}(X)$. By Lemma 7.3, A is clopen in $(X, \tau, \leq)$. Since $\mathsf{CpUp}(X) \cup \mathsf{CpDo}(X)$ is a subbasis for τ and A is compact in (X, τ), we have $A = (U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$ for some $U_1, \ldots, U_n \in \mathsf{CpUp}(X)$ and $V_1, \ldots, V_n \in \mathsf{CpDo}(X)$. By Proposition 3.4, $\mathsf{CpUp}(X) = \beta_1$ and $\mathsf{CpDo}(X) = \beta_2$. Therefore, for each $i = 1, \ldots, n$ we have $\mathrm{Cl}_1(U_i \cap V_i) \in \beta_2$. Thus, $\mathrm{Cl}_1(A) = \mathrm{Cl}_1[(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)] = \mathrm{Cl}_1(U_1 \cap V_1) \cup \cdots \cup \mathrm{Cl}_1(U_n \cap V_n) \in \beta_2 = \mathsf{CpDo}(X)$. This implies that $\mathrm{Cl}_1(A)$ is clopen in $(X, \tau, \leq)$, so by Lemma 7.3, $\mathrm{Cl}_1(A) \in \mathsf{PC}(X)$, and so (X, τ_1, τ_2) is a bitopological Esakia space. $\dashv$

From now on we will call a pairwise Stone space a bitopological Esakia space if it satisfies the condition of Theorem 7.5.

Theorem 7.6. *Let $(X, \tau, \leq)$ be a Priestley space and (X, τ_1, τ_2) be the corresponding pairwise Stone space. Then $(X, \tau, \leq)$ is an Esakia space iff (X, τ_1, τ_2) is a bitopological Esakia space.*

Proof. Since $\mathsf{Cp}(X) = \mathsf{PC}(X)$ and for $A \in \mathsf{PC}(X)$ we have $\mathrm{Cl}_1(A) = {\downarrow}A$, the result follow. $\dashv$

In order to characterize morphisms between bitopological Esakia spaces, we recall the following characterization of p-morphisms.

Lemma 7.7. [10, pp. 17-18] *For two posets $(X, \leq)$ and $(X', \leq')$ and a map $f : X \to X'$, the following conditions are equivalent:*

(1) *f is a p-morphism.*
(2) *For each $x \in X$ we have $f({\uparrow}x) = {\uparrow}f(x)$.*
(3) *For each $x' \in X'$ we have $f^{-1}({\downarrow}x') = {\downarrow}f^{-1}(x')$.*

Definition 7.8. *Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be two bitopological Esakia spaces. We call a map $f : X \to X'$ a* bitopological Esakia morphism *if f is bi-continuous and $f(\mathrm{Cl}_2(x)) = \mathrm{Cl}_2'(f(x))$ for each $x \in X$.*

Let $(X, \tau, \leq)$ and $(X', \tau', \leq')$ be two Esakia spaces, (X, τ_1, τ_2) and (X', τ_1', τ_2') be the corresponding bitopological Esakia spaces, and $f : X \to X'$ be bi-continuous. By Corollary 6.6, for each $x \in X$ we have $\uparrow x = \mathrm{Cl}_2(x)$ and $\downarrow x = \mathrm{Cl}_1(x)$. Therefore, by Lemma 7.7, f is an Esakia morphism iff f is a bitopological Esakia morphism iff $f^{-1}(\mathrm{Cl}_1(x')) = \mathrm{Cl}_1(f^{-1}(x'))$.

Let **BEsa** denote the category of bitopological Esakia spaces and bitopological Esakia morphisms. Clearly **BEsa** is a proper subcategory of **PStone**. Moreover, putting the results obtained above together, we obtain:

Theorem 7.9. *The categories* **Esa** *and* **BEsa** *are isomorphic. Consequently,* **Heyt** *is dually equivalent to* **BEsa**.

Let (X, τ) be a spectral space. We call $Y \subseteq X$ a *doubly spectral subset* of (X, τ) if both Y and Y^c are spectral subsets of (X, τ). Let $\mathsf{DS}(X)$ denote the set of doubly spectral subsets of X.

Definition 7.10. *Let (X, τ) be a spectral space. We call (X, τ) a* spectral Esakia space *if $A \in \mathsf{DS}(X)$ implies $\mathrm{Cl}(A) \in \mathsf{DS}(X)$.*

Theorem 7.11. *Let (X, τ_1, τ_2) be a pairwise Stone space and (X, τ_1) be the corresponding spectral space. Then (X, τ_1, τ_2) is a bitopological Esakia space iff (X, τ_1) is a spectral Esakia space.*

Proof. By Lemma 7.3, $\mathsf{PC}(X) = \mathsf{DS}(X)$. The result follows. ⊣

For two spectral Esakia spaces (X, τ) and (X', τ'), we call a map $f : X \to X'$ a *spectral Esakia morphism* if f is spectral and $f(\mathrm{Sat}(x)) = \mathrm{Sat}'(f(x))$.

Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be two bitopological Esakia spaces and (X, τ_1) and (X', τ_1') be the corresponding spectral Esakia spaces. By Corollary 6.6, for each $x \in X$ we have $\mathrm{Cl}_2(x) = \mathrm{Sat}_1(x)$ and $\mathrm{Cl}_1(x) = \mathrm{Sat}_2(x)$. Therefore, a bi-continuous map $f : X \to X'$ is a bi-Esaki morphism iff f is a spectral Esakia morphism iff $f^{-1}(\mathrm{Cl}_1(x')) = \mathrm{Cl}_1(f^{-1}(x'))$.

Let **SpecE** denote the category of spectral Esakia spaces and spectral Esakia morphisms. Clearly **SpecE** is a proper subcategory of **Spec**. Moreover, putting the results obtained above together, we obtain:

Theorem 7.12. *The categories* **Esa**, **BEsa**, *and* **SpecE** *are isomorphic. Consequently,* **Heyt** *is also dually equivalent to* **SpecE**.

Remark 7.13. In Remark 5.4 we pointed out that the duality between **DLat** and the categories **Pries**, **PStone**, and **Spec** can be obtained through the schizophrenic object **2**. On the other hand, there is no schizophrenic object that induces the duality for Heyting algebras. To see this, let there exist a schizophrenic object S in **Heyt** such that the duality between **Heyt** and, say, **Esa** is obtained through S. Then S is also an object of **Esa** and the functors $(-)_* : \mathbf{Heyt} \to \mathbf{Esa}$ and $(-)^* : \mathbf{Esa} \to \mathbf{Heyt}$ can be described through S; that is, for each object A of **Heyt**, the carrier of A_* is the set $\mathrm{Hom}_{\mathbf{Heyt}}(A, S)$ and for each object X of **Esa**, the carrier of X^* is the set $\mathrm{Hom}_{\mathbf{Esa}}(X, S)$. Therefore, the isomorphism $\varphi : A \to {A_*}^*$ is given by $\varphi(a)(h) = h(a)$ for each $a \in A$ and $h \in A_*$. Thus, if $a \neq b$ in A, then there exists $h \in \mathrm{Hom}_{\mathbf{Heyt}}(A, S)$ such that $h(a) \neq h(b)$. We show that this leads to a contradiction. Let A be a linearly ordered Heyting algebra with the second largest element a. Then $a \neq 1$. We observe that each $h \in \mathrm{Hom}_{\mathbf{Heyt}}(A, S)$ for which $h(a) \neq 1$ is injective. Indeed, let $b < c \leq a$. If $h(b) = h(c)$, then $h(b) = h(c \to b) = h(c) \to h(b) = 1$. This together with $h(b) \leq h(a)$ imply $h(a) = 1$, a contradiction. Consequently, such an S cannot exist because it would contain a subset of an arbitrarily large cardinality. Clearly this argument does not depend on the category **Esa**. In fact, it shows that there is no *co-generating* object in **Heyt**, and hence the duality for Heyting algebras can not be induced by a schizophrenic object. For a general discussion of co-generators and dualities which are obtained through schizophrenic objects we refer to Johnstone [17, p. 254].

The dual description of algebraic concepts important for the study of Heyting algebras is similar to that of bounded distributive lattices. The dual description of filters, prime filters, and maximal filters as well as ideals, prime ideals, and maximal ideals is exactly the same. So is the dual description of the canonical completions. On the other hand, the dual description of the MacNeille completions gets simplified [15, Sec. 3] because in the case of Heyting algebras, we have $\mathbf{D} = \mathrm{Cl}$.

It is well-known that homomorphic images of a Heyting algebra A are characterized by its filters. Consequently, unlike the case of bounded distributive lattices, homomorphic images of a Heyting algebra A dually correspond to closed upsets of the Esakia space of A. Therefore, homomorphic images of A dually correspond to τ_2-closed subsets of the bitopological Esakia space of A, and to compact saturated subsets of the spectral Esakia space of A.

We give the dual description of subalgebras of a Heyting algebra. For a quasi-ordered set (X, Q), we define an equivalence relation E on X by xEy iff xQy and yQx.

Definition 7.14. *Let $(X, \tau, \leq)$ be a Priestley space and Q be a Priestley quasi-order on X extending $\leq$. We call Q an* Esakia quasi-order *if for each $x, y \in X$, from xQy it follows that there exists $z \in X$ such that $x \leq z$ and zEy.*

Remark 7.15. Let $(X, \tau, \leq)$ be a Priestley space and E be an equivalence relation on X. We call E an *Esakia equivalence relation* if E viewed as a quasi-order is a Priestley quasi-order on X and ${\uparrow}E(x) \subseteq E({\uparrow}x)$. It is easy to see that if Q is an Esakia quasi-order, then E is an Esakia equivalence relation. For an Esakia equivalence relation E, we define Q on X by xQy iff there exists $z \in X$ such that $x \leq z$ and zEy . Then for an Esakia space X, it is easy to see that Q is an Esakia quasi-order. Thus, for an Esakia space X, there is an isomorphism between Esakia quasi-orders on X ordered by inclusion and Esakia equivalence relations on X ordered by inclusion.

Theorem 7.16. *Let A be a Heyting algebra and $(X, \tau, \leq)$ be the Esakia space of A. Then the poset $(\mathsf{HS}_A, \subseteq)$ of Heyting subalgebras of A is dually isomorphic to the poset $(\mathsf{EQ}_X, \subseteq)$ of Esakia quasi-orders on X.*

Proof. In view of Theorem 6.15, it is sufficient to show that if $S \in \mathsf{HS}_A$, then $Q_S \in \mathsf{EQ}_X$, and that if $Q \in \mathsf{EQ}_X$, then $S_Q \in \mathsf{HS}_A$. Let $S \in \mathsf{HS}_A$. By Theorem 6.15, Q_S is a Priestley quasi-order on X extending $\leq$. Suppose that xQ_Sy. Then $x \cap S \subseteq y \cap S$. Let F be the filter of A generated by $x \cup (y \cap S)$. Then F is a proper filter of A with $x \subseteq F$ and $F \cap S = y \cap S$. By Zorn's lemma we can extend F to a maximal such filter z. The standard argument shows that z is prime. Therefore, there exists $z \in X$ such that $x \leq z$ and zE_Sy. Thus, $Q_S \in \mathsf{EQ}_X$. Now let $Q \in \mathsf{EQ}_X$. By Theorem 6.15, S_Q is a bounded distributive sublattice of A. For $a, b \in S_Q$ we have $\phi(a), \phi(b)$ are Q-upsets of X. We show that $\phi(a \to b) = \phi(a) \to \phi(b) = [{\downarrow}(\phi(a) - \phi(b))]^c = \{x \in X \mid {\uparrow}x \cap \phi(a) \subseteq \phi(b)\}$ is also a Q-upset of X. Let $x \in \phi(a \to b)$ and xQy. We show that ${\uparrow}y \cap \phi(a) \subseteq \phi(b)$. Let $u \in {\uparrow}y \cap \phi(a)$. Then $y \leq u$ and $u \in \phi(a)$. Therefore, xQu, and so there exists $z \in X$ such that $x \leq z$ and zEu. Since zEu, $u \in \phi(a)$, and $\phi(a)$ is a Q-upset, we have $z \in \phi(a)$. This implies that $z \in {\uparrow}x \cap \phi(a)$ an as ${\uparrow}x \cap \phi(a) \subseteq \phi(b)$, we obtain $z \in \phi(b)$. Now zEu and $\phi(b)$ being a Q-upset imply that $u \in \phi(b)$. Consequently, ${\uparrow}y \cap \phi(a) \subseteq \phi(b)$, so $y \in \phi(a \to b)$, and so $\phi(a \to b)$ is a Q-upset. It follows that $a, b \in S_Q$ implies $a \to b \in S_Q$, and so $S_Q \in \mathsf{HS}_A$. $\dashv$

As a consequence of Remark 7.15 and Theorem 7.16, we obtain the following well-known dual description of subalgebras of Heyting algebras [8, Thm. 4]: The poset of Heyting subalgebras of a Heyting algebra A is dually isomorphic to the poset of Esakia equivalence relations on the Esakia space X of A.

Now we give the dual description of subalgebras of Heyting algebras by means of bitopological Esakia spaces and spectral Esakia spaces. Let (X, τ_1, τ_2) be a bitopological Esakia space. We call a bitopology (τ_1', τ_2') an *Esakia bitopology* on X if (τ_1', τ_2') is pairwise zero-dimensional and $A \in \beta_1', B \in \beta_2'$ imply $\mathrm{Cl}_1(A \cap B) \in \beta_2'$. Let $(\mathsf{EB}_X, \subseteq)$ denote the poset of Esakia bitopologies on X coarser than (τ_1, τ_2).

Lemma 7.17. *Let $(X, \tau, \leq)$ be an Esakia space and (X, τ_1, τ_2) be the corresponding bitopological Esakia space. Then $(\mathsf{EQ}_X, \subseteq)$ is dually isomorphic to $(\mathsf{EB}_X, \subseteq)$.*

Proof. In view of Lemma 6.17, we only need to show that if $Q \in \mathsf{EQ}_X$, then $(\tau_1^Q, \tau_2^Q) \in \mathsf{EB}_X$, and that if $(\tau_1', \tau_2') \in \mathsf{EB}_X$, then $Q_{(\tau_1', \tau_2')} \in \mathsf{EQ}_X$. Let $Q \in \mathsf{EQ}_X$. By Lemma 6.17, (τ_1^Q, τ_2^Q) is a zero-dimensional bitopology coarser than (τ_1, τ_2). Moreover, β_1^Q coincides with the set of clopen Q-upsets and β_2^Q coincides with the set of clopen Q-downsets of $(X, \tau, \leq)$. Therefore, for $A \in \beta_1^Q$ and $B \in \beta_2^Q$ we have that A is a clopen Q-upset and B is a clopen Q-downset of $(X, \tau, \leq)$. Since Q is an Esakia quasi-order, by Theorem 7.16, the lattice of clopen Q-upsets of $(X, \tau, \leq)$ is a Heyting subalgebra of the Heyting algebra of all clopen upsets of $(X, \tau, \leq)$. Thus, ${\downarrow}(A \cap B)$ is a clopen Q-downset of $(X, \tau, \leq)$, and so ${\downarrow}(A \cap B) \in \beta_2^Q$. By Corollary 6.6, $\mathrm{Cl}_1(A \cap B) = {\downarrow}(A \cap B)$. Consequently, $\mathrm{Cl}_1(A \cap B) \in \beta_2^Q$, and so $(\tau_1^Q, \tau_2^Q) \in \mathsf{EB}_X$. Now suppose that $(\tau_1', \tau_2') \in \mathsf{EB}_X$. By Lemma 6.17, $Q_{(\tau_1', \tau_2')}$ is a Priestley quasi-order on X extending $\leq$. We show that the lattice of clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$ is closed under $\to$. Let A and B be clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$. Then $A \in \beta_1'$ and $B^c \in \beta_2'$. Therefore, $\mathsf{Cl}_1(A \cap B^c) \in \beta_2'$, and so $\mathsf{Cl}_1(A \cap B^c)$ is a clopen $Q_{(\tau_1', \tau_2')}$-downset of $(X, \tau, \leq)$. By Corollary 6.6, $\mathrm{Cl}_1(A \cap B^c) = {\downarrow}(A \cap B^c)$. Consequently, ${\downarrow}(A \cap B^c)$ is a clopen $Q_{(\tau_1', \tau_2')}$-downset of $(X, \tau, \leq)$, so $A \to B = [{\downarrow}(A \cap B^c)]^c$ is a clopen $Q_{(\tau_1', \tau_2')}$-upset of $(X, \tau, \leq)$, and so the lattice of clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$ is closed under $\to$. This implies that the lattice of clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$ is a Heyting subalgebra of the Heyting algebra of all clopen upsets of $(X, \tau, \leq)$, which, by Theorem 7.16, gives us that $Q_{(\tau_1', \tau_2')} \in \mathsf{EQ}_X$. ⊣

Let (X, τ) be a spectral Esakia space. We call a topology τ' on X a *spectral Esakia topology* if τ' is strongly coherent and $A \in \mathcal{E}(X, \tau'), B \in \Delta(X, \tau')$ imply $\mathrm{Cl}(A \cap B) \in \Delta(X, \tau')$. For a spectral Esakia space (X, τ), let $(\mathsf{SE}_X, \subseteq)$ denote the poset of spectral Esakia topologies on X coarser than τ.

Lemma 7.18. *Let (X, τ_1, τ_2) be a bitopological Esakia space and (X, τ_1) be the corresponding spectral Esakia space. Then $(\mathsf{EB}_X, \subseteq)$ is isomorphic to $(\mathsf{SE}_X, \subseteq)$.*

Proof. In view of Lemma 6.19, we only need to show that if $(\tau_1', \tau_2') \in \mathsf{EB}_X$, then $\tau_1' \in \mathsf{SE}_X$, and that if $\tau_1' \in \mathsf{SE}_X$, then $(\tau_1', \tau_2') \in \mathsf{EB}_X$. Let $(\tau_1', \tau_2') \in \mathsf{EB}_X$. By Lemma 6.19, τ_1' is a strongly coherent topology coarser than τ_1. Moreover, since $\beta_1' = \mathcal{E}(X, \tau_1')$ and $\beta_2' = \Delta(X, \tau_1')$, for $A \in \mathcal{E}(X, \tau_1')$ and $B \in \Delta(X, \tau_1')$, we have $A \in \beta_1'$ and $B \in \beta_2'$, so $\mathrm{Cl}_1(A \cap B) \in \beta_2'$, and so $\mathrm{Cl}_1(A \cap B) \in \Delta(X, \tau_1')$. Therefore, $\tau_1' \in \mathsf{SE}_X$. Now let $\tau_1' \in \mathsf{SE}_X$. By Lemma 6.19, (τ_1', τ_2') is a zero-dimensional bitopology coarser than (τ_1, τ_2). Moreover, since $\mathcal{E}(X, \tau_1') = \beta_1'$ and $\Delta(X, \tau_1') = \beta_2'$, for $A \in \beta_1'$ and $B \in \beta_2'$, we have $A \in \mathcal{E}(X, \tau_1')$ and $B \in \Delta(X, \tau_1')$, so $\mathrm{Cl}_1(A \cap B) \in \Delta(X, \tau_1')$, and so $\mathrm{Cl}_1(A \cap B) \in \beta_2'$. Thus, $(\tau_1', \tau_2') \in \mathsf{EB}_X$. ⊣

Heyt	**Esa**	**BEsa**	**SpecE**
filter	closed upset	τ_2-closed set	compact saturated set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
ideal	open upset	τ_1-open set	open set
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	closed upset	τ_2-closed set	compact saturated set
subalgebra	$Q \in \mathsf{EQ}_X$	$(\tau_1', \tau_2') \in \mathsf{EB}_X$	$\tau' \in \mathsf{SE}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille completion	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

Table 2. Dictionary for **Heyt**, **Esa**, **BEsa**, and **SpecE**.

Putting Lemmas 7.17 and 7.18 together, we obtain the following dual description of Heyting subalgebras of a Heyting algebra.

Corollary 7.19. *Let A be a Heyting algebra, $(X, \tau, \leq)$ be the Esakia space of A, (X, τ_1, τ_2) be the bitopological Esakia space of A, and (X, τ_1) be the spectral Esakia space of A. Then $(\mathsf{HS}_A, \subseteq)$ is dually isomorphic to $(\mathsf{EQ}_X, \subseteq)$, and is isomorphic to $(\mathsf{EB}_X, \subseteq)$ and $(\mathsf{SE}_X, \subseteq)$.*

In Table 2 we gather together the dual descriptions of different algebraic concepts for Heyting algebras by means of their Esakia spaces, bitopological Esakia spaces, and spectral Esakia spaces obtained in this section. This can be thought of as a dictionary of duality theory for Heyting algebras.

We conclude by mentioning that two more natural subclasses of the class of distributive lattices that play an important role in the study of non-classical logics are the classes of co-Heyting algebras and bi-Heyting algebras, respectively. We recall that a *co-Heyting algebra* is a bounded distributive lattice A with a binary operation $\leftarrow: A^2 \to A$ such that for all $a, b, c \in A$ we have:

$$c \geq a \leftarrow b \text{ iff } b \vee c \geq a.$$

We also recall that $(A, \to, \leftarrow)$ is a *bi-Heyting algebra* if $(A, \to)$ is a Heyting algebra and $(A, \leftarrow)$ is a co-Heyting algebra. The first duality for co-Heyting algebras and bi-Heyting algebras was developed by Esakia [9]. It is a restricted version of Priestley's duality, and is a modified version of Esakia's duality for Heyting algebras [8]. The bitopological and spectral dualities for co-Heyting and bi-Heyting algebras can be developed by an obvious modification of the bitopological and spectral dualities for Heyting algebras developed in this section. We skip the details, which can be recovered by an appropriate modification of the proofs given above, and only mention that there is a dictionary of duality theory for co-Heyting algebras and bi-Heyting algebras similar to the one for Heyting algebras given in Table 2.

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