Coinduction - An Introductory Example

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based on: Jacobs and Rutten, A Tutorial on (Co)Algebras and (Co)Induction. EATCS Bulletin 62, 1997, p.222-259

coinduction

How to reason about infinite data/processes? An example:

Given $zip : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$, $even : A^{\mathbb{N}} \to A^{\mathbb{N}}$, $odd : A^{\mathbb{N}} \to A^{\mathbb{N}}$

such that

$$\begin{aligned} head(zip(l_1, l_2)) &= head(l_1) \\ tail(zip(l_1, l_2)) &= zip(l_2, tail(l_1)) \end{aligned}$$

$$\begin{aligned} head(even(l)) &= head(l) \\ tail(even(l)) &= even(tail(tail(l))) \end{aligned}$$

$$odd(l) = even(tail(l))$$

show

$$zip(even(x), odd(x)) = x$$

example, cont'd

Given

$$head(zip(l_1, l_2)) = head(l_1)$$
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$$head(even(l)) = head(l)$$
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$$zip(even(x), odd(x))) = x$$

$$head(zip(even(x), odd(x))) \;=\; head(even(x)) \;=\; head(x)$$

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$$zip(even(x), odd(x))) = x$$

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example, cont'd

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explanation (bisimulation)

Coinduction Proof Principle: Two streams $x, x' \in A^{\mathbb{N}}$ are equal iff there is a relation R with xRx' and for all y, y'

$$\begin{aligned} yRy' &\Rightarrow head(y) = head(y') \\ yRy' &\Rightarrow tail(y)\ R\ tail(y') \end{aligned}$$

Example: Put $zip(even(x), odd(x))\ R\ x$ for all $x \in A^{\mathbb{N}}$

$$\begin{aligned} tail(zip(even(x), odd(x))) &= zip(odd(x), tail(even(x))) \\ &= zip(odd(x), even(tail(tail(x)))) \\ &= zip(even(tail(x)), odd(tail(x)))\quad R\quad tail(x) \end{aligned}$$

explanation (coinduction via finality)

Def: An object Z is **final** if for all objects X there is a unique arrow $X \to Z$.

Observation: $A^{\mathbb{N}} \to A \times A^{\mathbb{N}}$ is the final coalgebra (for the functor $TX = A \times X$).

Fact: To say that $X \to A \times X$ satisfies the coinduction proof principle is equivalent to saying that $X \to A \times X$ is the final coalgebra.

why category theory matters

. . . simple elegant definitions (via universal properties, eg finality)

. . . category theoretic definitions are more general

. . . the right level of abstraction for many proofs (eg Lambek's lemma, Birkhoff's variety theorem)

. . . solution of domain equations

. . . duality

. . . heuristics for finding meaningful mathematical constructions

. . . CT often codes up a lot of annoying combinatorics (eg: GSOS rule format is equivalent to the naturality of a transformation between two functors)