

# On Universal Algebra over Nominal Sets

Alexander Kurz, Daniela Petrişan

*Department of Computer Science, University of Leicester, UK*

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We investigate universal algebra over the category  $\mathbf{Nom}$  of nominal sets. Using the fact that  $\mathbf{Nom}$  is a full reflective subcategory of a monadic category, we obtain an HSP-like theorem for algebras over nominal sets. We isolate a ‘uniform’ fragment of our equational logic, which corresponds to the nominal logics present in the literature. We give semantically invariant translations of theories for nominal algebra and NEL into ‘uniform’ theories and systematically prove HSP theorems for models of these theories.

## 1. Introduction

Nominal sets were introduced by Gabbay and Pitts (Gabbay and Pitts, 1999). This paper describes a step towards universal algebra over nominal sets. There has been some work in this direction, most notably by M.J. Gabbay (Gabbay, 2008). The originality of our approach is that we do not start from the analogy between sets and nominal sets. As shown in (Gabbay, 2008), this is possible, but it requires ingenuity and ad hoc constructions. For example, the logic of (Gabbay, 2008) is not standard equational logic and even fundamental notions such as variables and free algebras have to be revisited.

In contrast, we want to explore how far one can get by using completely standard universal algebra. This is based on the well-known observation that  $\mathbf{Nom}$  is equivalent to a full reflective subcategory  $\mathbf{Sh}(\mathbb{I}^{op})$  of the presheaf category  $\mathbf{Set}^{\mathbb{I}}$ , where  $\mathbb{I}$  is the category of finite subsets of a countable set of names  $\mathcal{N}$ , with injective maps.

$$\mathbf{Sh}(\mathbb{I}^{op}) \begin{array}{c} \longleftarrow \\ \perp \\ \longrightarrow \end{array} \mathbf{Set}^{\mathbb{I}} \quad (1)$$

Like any presheaf category,  $\mathbf{Set}^{\mathbb{I}}$  is a category of many-sorted algebras: the sorts being the objects of  $\mathbb{I}$ , the (unary) operations the arrows in  $\mathbb{I}$ , and the equations the commuting diagrams of  $\mathbb{I}$ . Thus, diagram (1) embeds nominal sets into an equationally defined class of many-sorted algebras. To do the same for algebras, we need a notion of signature. This is provided by endofunctors  $\tilde{L} : \mathbf{Sh}(\mathbb{I}^{op}) \rightarrow \mathbf{Sh}(\mathbb{I}^{op})$  and  $L : \mathbf{Set}^{\mathbb{I}} \rightarrow \mathbf{Set}^{\mathbb{I}}$ . The intention here is that the two functors  $\tilde{L}$  and  $L$  are “doing the same”, or, more precisely they commute with the left adjoint. Then the adjunction (1) lifts to algebras:

$$\begin{array}{ccc}
 \text{Alg}(\tilde{L}) & \begin{array}{c} \xleftarrow{\bar{R}} \\ \perp \\ \xrightarrow{\bar{I}} \end{array} & \text{Alg}(L) \\
 \downarrow \tilde{U} & & \downarrow U \\
 \tilde{L} \circlearrowleft \text{Sh}(\mathbb{I}^{op}) & \begin{array}{c} \xleftarrow{R} \\ \perp \\ \xrightarrow{I} \end{array} & \text{Set}^{\mathbb{I}} \circlearrowright L
 \end{array} \tag{2}$$

At this point, we are encouraged by a result of (Kurz and Rosický, 2006), stating that all ‘universal-algebraic’ endofunctors  $L$  have a presentation. In particular, it follows that  $\text{Alg}(L)$  is an equationally defined class (variety) of many-sorted algebras *in the standard sense of universal algebra*. For example, we have an equational calculus at hand, which we can use to reason about algebras in  $\text{Alg}(L)$  and, hence, to reason about algebras in the full subcategory  $\text{Alg}(\tilde{L})$ .

Central to our approach is the flexibility provided by the notion of a presentation of functor (introduced in (Bonsangue and Kurz, 2006) and further developed in (Kurz and Rosický, 2006; Kurz and Petrişan, 2008)). It allows us to identify the abstract notion of  $L$ -algebras with the concrete notion of algebras for a signature and equations. In particular, we may prove results about equational logic in an abstract syntax free manner. Thus Diagram (2) suggests a program of developing universal algebra over nominal sets and we will carry out the following first steps.

Section 3 proves an HSP-like theorem for algebras over  $\text{Sh}(\mathbb{I}^{op})$ . We have Birkhoff’s HSP-theorem for  $\text{Alg}(L)$  and using a category theoretic formulation of Birkhoff’s proof (which is standard, see e.g. (Adámek et al., 1990, Exercise 16G)), it is straightforward to transfer it to the nominal setting: a class of  $\tilde{L}$ -algebras is equationally definable if and only if it is closed under products, subalgebras and presheaf-quotients. The usefulness of changing the perspective from  $\text{Nom}$  to  $\text{Set}^{\mathbb{I}}$  is highlighted by Gabbay’s remark in (Gabbay, 2008) that “An attempt to directly transfer proofs of the HSP theorem to the nominal setting fails”.

Section 4 extracts from the abstract category theoretic treatment of Section 3 a standard equational calculus. To this end we give presentations by operations and equations of  $\text{Set}^{\mathbb{I}}$  and the ‘shift’ or ‘abstraction’ functor  $\delta$ . The functor  $\delta$  satisfies the conditions set out in Section 3 and we obtain an HSP theorem for algebras with binders. We illustrate the equational calculus with an axiomatisation of the  $\lambda$ -calculus. The observation that this axiomatisation is ‘uniform’ over the indexing sorts in  $\mathbb{I}$  leads to the next section.

Section 5 starts from the observation that the equational logic of the previous sections is too strong in that it does not need to respect a fundamental notion of nominal sets, namely equivariance. We therefore introduce a fragment of our equational logic, called the uniform fragment. We prove an HSPA theorem in the style of (Gabbay, 2008): a class of  $\tilde{L}$ -algebras is definable by uniform equations if and only if it is closed under

quotients, subalgebras, products and abstraction. This shows that the uniform fragment of equational logic has the same expressiveness as the nominal algebra of (Gabbay, 2008).

Section 6 shows how to translate theories of nominal algebra (Gabbay and Mathijssen, 2009) and nominal equational logic (Clouston and Pitts, 2007) into uniform theories. We prove that these translations are semantically invariant and we obtain an HSPA theorem for nominal equational logic and a new proof of the HSPA theorem of nominal algebra (Gabbay, 2008).

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## 2. Preliminaries

### 2.1. Sheaves and Presheaves on $\mathbb{I}^{op}$

For the remainder of this article we use  $\mathcal{N}$  to denote an infinite countable set of names. We will denote by  $\mathbb{I}$  the category whose objects are finite subsets of  $\mathcal{N}$  and whose morphisms are injective maps. A presheaf on  $\mathbb{I}^{op}$  is an object in the functor category  $\mathbf{Set}^{\mathbb{I}}$ , that is, a functor from  $\mathbb{I}$  to  $\mathbf{Set}$ . Note that  $\mathbf{Set}^{\mathbb{I}}$  is a category of many-sorted algebras over  $\mathbf{Set}^{\mathcal{P}_f(\mathcal{N})}$ , where  $\mathcal{P}_f(\mathcal{N})$  denotes the set of finite subsets of  $\mathcal{N}$ . The unary operations are the arrows in  $\mathbb{I}$ , and the equations are the commuting diagrams of  $\mathbb{I}$ .

The category of interest throughout this paper is the full subcategory of  $\mathbf{Set}^{\mathbb{I}}$  consisting of pullback preserving functors. This category is also known in the literature as the *Schanuel topos* and is denoted by  $\mathbf{Sh}(\mathbb{I}^{op})$ . The notation is justified by the fact that  $\mathbf{Sh}(\mathbb{I}^{op})$  is a Grothendieck topos: a functor  $A : \mathbb{I} \rightarrow \mathbf{Set}$  preserves pullbacks if and only if it is a sheaf for the atomic topology on  $\mathbb{I}^{op}$ , (Johnstone, 2002). We will list a few properties of this category relevant for the development of this paper.

**Proposition 2.1.** The inclusion functor  $I : \mathbf{Sh}(\mathbb{I}^{op}) \rightarrow \mathbf{Set}^{\mathbb{I}}$  has a left adjoint  $R$ , which preserves finite limits.

This is a particular instance of (Maclane and Moerdijk, 1994, Theorem 1, pp.128).

**Remark 2.2.** The functor  $I$  preserves filtered colimits and coproducts. This is because both filtered colimits and coproducts commute with pullbacks in  $\mathbf{Set}$ . Also  $I$  preserves all limits and  $R$  preserves all colimits. This follows from the fact that  $I$  is right adjoint to  $R$ .

**Proposition 2.3.** A morphism between two presheaves is an epimorphism if and only if it is so pointwise. A sheaf morphism  $f : A \rightarrow B$  is an epimorphism in  $\mathbf{Sh}(\mathbb{I}^{op})$  if and only if for all finite sets of names  $S$  and all  $y \in B(S)$  there exists an inclusion  $l : S \rightarrow T$  in  $\mathbb{I}$  and  $x \in A(T)$  such that  $f_T(x) = B(l)(y)$ .

*Proof.* This follows from (Maclane and Moerdijk, 1994, Corollary III.7.5.).  $\square$

If  $A$  is a presheaf, we will call  $RA$  the *sheafification* of  $A$ .

**Remark 2.4.** Starting with a presheaf  $A$ , the sheaf  $RA$  is obtained as follows. For finite sets of names  $S \subseteq T$ , we will say that an element  $x \in A(T)$  is supported by  $S$  iff whenever two functions  $i, j : T \rightarrow T'$  coincide when restricted to  $S$ , then  $A(i)(x) = A(j)(x)$ . Two elements supported by  $S$ ,  $x \in A(T)$  and  $y \in A(T')$ , are said to be equivalent iff there exist injective maps  $i : T \rightarrow T''$  and  $j : T' \rightarrow T''$ , such that  $i(s) = j(s)$  for all  $s \in S$  and such that  $A(i)(x) = A(j)(y)$ . We denote by  $\bar{x}$  the equivalence class of  $x$ . We define  $RA(S)$  to be the set of equivalence classes of elements supported by  $S$ . If  $i : S \rightarrow S'$  is an injective map, then  $RA(i)$  maps  $\bar{x}$  to  $\overline{A(j)(x)}$ , where  $j$  is a map that makes the square

$$\begin{array}{ccc} S & \xrightarrow{i} & S' \\ \downarrow w & & \downarrow w' \\ T & \xrightarrow{j} & T' \end{array} \quad (3)$$

commute for appropriate inclusions  $w, w'$ .

## 2.2. Finitary Presentations for Functors on Many-sorted Varieties

Let  $S$  be a set of sorts. A *signature* is a set of operation symbols together with an arity map. For each signature we consider an endofunctor on  $\mathbf{Set}^S$ , such that for  $X \in \mathbf{Set}^S$  we define  $\Sigma X \in \mathbf{Set}^S$  by

$$\Sigma X(s) = \coprod \{f_{s_1 \times \dots \times s_n \rightarrow s}\} \times X(s_1) \times \dots \times X(s_n) \quad (4)$$

where the coproduct is taken after all operation symbols  $f$  whose result has sort  $s$ . The algebras for a signature are precisely the algebras for the corresponding endofunctor, and form the category denoted by  $\mathbf{Alg}(\Sigma)$ . The terms over an  $S$ -sorted set of variables  $X$  are defined in the standard manner and an equation consists of a pair  $(\tau_1, \tau_2)$  of terms of the same sort, usually denoted by  $\tau_1 = \tau_2$ . A  $\Sigma$ -algebra  $A$  satisfies this equation if and only if, for any interpretation of the variables of  $X$ , we obtain equality in  $A$ . A full subcategory  $\mathcal{A}$  of  $\Sigma$ -algebras is called an *equational class* or *variety* if there exists a set of equations  $E$  such that an algebra lies in  $\mathcal{A}$  if and only if it satisfies all the equations of  $E$ . Such an equational class will be denoted by  $\mathbf{Alg}(\Sigma, E)$ .

The notion of functors with finitary presentations has been introduced in (Bonsangue and Kurz, 2006) for applications to coalgebraic logic. Let  $\mathcal{A} = \mathbf{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$  be an equational class. Denote by  $U : \mathcal{A} \rightarrow \mathbf{Set}^S$  the forgetful functor and by  $F$  its left adjoint. Intuitively an endofunctor  $L$  on  $\mathcal{A}$  has a finitary presentation by operations  $\Sigma_L$  and equations  $E_L$  if, for each object  $A$ , we have that  $LA$  is uniformly isomorphic to a quotient of  $F\Sigma_L UA$  by the equations  $E_L$ , in a sense that will be made precise in the next definition:

**Definition 2.5.** A finitary presentation for an endofunctor  $L$  on  $\mathcal{A}$  is a pair  $\langle \Sigma_L, E_L \rangle$  where  $\Sigma_L$  is an endofunctor as in (4) and  $E_L$  is a set of equations as follows: for any  $S$ -sorted set of variables  $V$ ,  $E_V$  is a subset of  $(UF\Sigma_L UFV)^2$  and  $E_L$  is the disjoint union of  $E_V$  taken over all finite sets of variables  $V$ . The functor  $L$  is presented by  $\langle \Sigma_L, E_L \rangle$  if, for any  $A \in \mathcal{A}$ , the algebra  $LA$  is the joint coequalizer:

$$FE_V \begin{array}{c} \xrightarrow{\pi_1^\sharp} \\ \xrightarrow{\pi_2^\sharp} \end{array} F\Sigma_L UFV \xrightarrow{F\Sigma_L Uv^\sharp} F\Sigma_L UA \xrightarrow{q_A} LA \quad (5)$$

which is taken over all finite sets of  $S$ -sorted variables  $V$  and all valuations  $v : V \rightarrow UA$ . Here  $v^\sharp$  denotes the adjoint transpose of a valuation  $v$ .

Functors with finitary presentations are characterised in (Kurz and Rosický, 2006) as exactly those functors which preserve sifted colimits. Recall that sifted colimits are exactly those colimits which commute with finite products in **Set**. Examples include filtered colimits and reflexive coequalizers.

For an endofunctor  $L$  on a category  $\mathcal{A}$ , we consider the category of  $L$ -algebras, denoted by  $\text{Alg}(L)$ , whose objects are defined as pairs  $(A, \alpha)$  such that  $\alpha : LA \rightarrow A$  is a morphism in  $\mathcal{A}$ . A morphism of  $L$ -algebras  $f : (A, \alpha) \rightarrow (A', \alpha')$  is a morphism  $f : A \rightarrow A'$  of  $\mathcal{A}$  such that  $f \circ \alpha = \alpha' \circ Lf$ .

If the functor  $L$  on an equational class has a finitary presentation, then  $\text{Alg}(L)$  is an equational class as well, and its presentation can be obtained in a modular way.

**Theorem 2.6.** Let  $\mathcal{A} = \text{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$  be an  $S$ -sorted equational class and let  $L : \mathcal{A} \rightarrow \mathcal{A}$  be a functor presented by operations  $\Sigma_L$  and equations  $E_L$ . Then  $\text{Alg}(L)$  is concretely isomorphic to  $\text{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$ .

Here by  $\Sigma_{\mathcal{A}} + \Sigma_L$  we understand the disjoint union of the signatures, and by  $E_{\mathcal{A}} + E_L$  the disjoint union of the equations  $E_{\mathcal{A}}$  and  $E_L$ , regarded as equations over  $\Sigma_{\mathcal{A}} + \Sigma_L$  (a proof of this theorem is in (Kurz and Petrişan, 2008)).

### 3. HSP Theorems

In this section we will prove an HSP-theorem for algebras over the topos  $\text{Sh}(\mathbb{I}^{op})$  in a systematic way.

In the first subsection we prove general results using categorical techniques. To set the scene we outline a categorical proof of Birkhoff's HSP theorem. Then, in Theorem 3.5, we show how to obtain an HSP-theorem for a full reflective subcategory  $\mathcal{A}$  of a category of algebras  $\mathcal{C}$ , if some additional conditions are met. Essentially, this is achieved by 'pushing' the proof of the general HSP-theorem through the adjunction

$$\mathcal{A} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathcal{C} .$$

This result is interesting because  $\mathcal{A}$  might not be a variety. We also prove a general result, Proposition 3.6, concerning a lifting property of an adjunction to categories of algebras for certain functors.

Secondly, in Section 3.2, we apply these results to the nominal setting. Consider diagram (2), where  $L$  is an endofunctor on  $\text{Set}^{\mathbb{I}}$  that preserves sifted colimits, and  $\tilde{L}$  is an endofunctor on  $\text{Sh}(\mathbb{I}^{op})$ , such that  $\tilde{L}R \simeq RL$ . Using Proposition 3.6 we prove that the

adjunction between  $\text{Sh}(\mathbb{I}^{op})$  and  $\text{Set}^{\mathbb{I}}$  can be lifted to an adjunction between  $\text{Alg}(\tilde{L})$  and  $\text{Alg}(L)$ . On the right hand side of this diagram we have categories monadic over  $\text{Set}^{\mathbb{N}}$ , for which the classical HSP-theorem holds. We derive an HSP theorem for  $\text{Alg}(\tilde{L})$ , by applying Theorem 3.5.

### 3.1. HSP theorems for full reflective subcategories

**Birkhoff’s HSP Theorem.** Given a category of algebras  $\mathcal{C}$ , a full subcategory  $\mathcal{B} \subseteq \mathcal{C}$  is closed under quotients (H for homomorphic images), subalgebras (S), and products (P) iff  $\mathcal{B}$  is definable by equations.

This theorem can be proved at different levels of generality. We assume here that  $\mathcal{C}$  is monadic over  $\text{Set}^{\kappa}$ , for some cardinal  $\kappa$ . We denote by  $U : \mathcal{C} \rightarrow \text{Set}^{\kappa}$  the forgetful functor and by  $F$  its left adjoint. Then we can identify a class of equations  $\Phi$  in variables  $X$  with quotients  $F X \rightarrow Q$ . Indeed, given  $\Phi$  we let  $Q$  be the quotient  $F X / \Phi$  and, conversely, given  $F X \rightarrow Q$  we let  $\Phi$  be the kernel of  $F X \rightarrow Q$ . Further, an algebra  $A \in \mathcal{C}$  satisfies the equations iff all  $F X \rightarrow A$  factor through  $F X \rightarrow Q$  as in the diagram

$$A \models \Phi \iff \begin{array}{ccc} F X & \xrightarrow{\quad} & F X / \Phi \\ & \searrow \vee & \swarrow \exists \\ & & A \end{array} \quad (6)$$

**Proof of ‘if’.** We want to show that a subcategory  $\mathcal{B}$  defined by equations  $\Phi$  is closed under HSP. Closure under subobjects  $A' \rightarrow A$  follows since quotients and subobjects form a factorisation system (see e.g. (Adámek et al., 1990, 14.1)). Indeed, according to (6), to show  $A \models \Phi \Rightarrow A' \models \Phi$  one has to find the dotted arrow in

$$\begin{array}{ccc} F X & \xrightarrow{\quad} & F X / \Phi \\ \downarrow & \swarrow \text{dotted} & \downarrow \\ A' & \xrightarrow{\quad} & A \end{array}$$

which exists because of the diagonal fill-in property of factorisation systems. A similar argument works for products (because of their universal property) and for quotients (using that free algebras are projective (Adámek et al., 1990, 9.27)).

**Proof of ‘only if’.** Given  $\mathcal{B} \subseteq \mathcal{C}$ , we first need to find the equations. Since  $\mathcal{B}$  is closed under SP,  $\mathcal{B}$  is a full reflective subcategory, that is, the inclusion  $\mathcal{B} \rightarrow \mathcal{C}$  has a left-adjoint  $H$  and, moreover, the unit  $A \rightarrow H A$  is a quotient.<sup>†</sup> We take as equations all  $F X \rightarrow H F X$ . That all  $A \in \mathcal{B}$  satisfy these equations, again using (6), follows immediately from the universal property of the left-adjoint  $H$ . Conversely, suppose that  $A$  satisfies all equations.

<sup>†</sup> To construct  $H A$  given  $A$ , consider all arrows  $f : A \rightarrow B_f$  with codomain in  $\mathcal{B}$ ; factor  $f = A \xrightarrow{q_f} \bar{B}_f \xrightarrow{i_f} B_f$ ; up to isomorphism, there is a only a proper set of different  $q_f$ ; now factor  $A \xrightarrow{\langle q_f \rangle} \prod_f \bar{B}_f$  as  $A \rightarrow H A \rightarrow \prod_f \bar{B}_f$  to obtain the unit  $A \rightarrow H A$ .

Consider the equations  $q : FUA \rightarrow HFUA$ . Because of (6), the counit  $e : FUA \rightarrow A$  must factor as  $e = f \circ q$ . Since  $e$  and  $q$  are quotients, so is  $f$ . Hence  $A$  is a quotient of  $HFUA$ , which is in  $\mathcal{B}$ .

**Remark 3.1.** Notice that in the proof above we allow quotients  $FX \rightarrow HFX$  for arbitrary  $\kappa$ -sorted sets  $X$ . If the set  $X$  is infinite, we allow equations involving infinitely many variables. Therefore we no longer reason within finitary logic. If we impose that the equations involve only finitely many variables, then the HSP theorem is *not* true for arbitrary many-sorted varieties. Indeed, in the many-sorted case, closure under homomorphic images, sub-algebras and products is no longer enough to deduce equational definability (see (Adámek et al., , Example 10.14.2)). One needs an additional constraint, namely closure under directed unions, (Adámek et al., , Theorem 10.12). But in the motivating examples of  $\mathbf{Set}^{\mathbb{I}}$ , we will prove that closure under HSP implies closure under directed unions.

In the following, we show that it is possible to obtain an HSP theorem for certain subcategories of varieties, by pushing the argument above through an adjunction. But first let us say what we mean in this context by **equationally definable** and **closed under HSP**. We will work in the following setting.

**Definition 3.2.** Let  $\mathcal{C}$  be a category monadic over  $\mathbf{Set}^{\kappa}$  for some cardinal  $\kappa$ , with  $U$  denoting the forgetful functor and  $F$  its left adjoint. Let  $\mathcal{A}$  be a full subcategory of  $\mathcal{C}$ , which has a factorisation system  $(E, M)$  such that morphisms in  $M$  are monomorphisms and the inclusion functor  $I$  has a left adjoint which preserves the regular factorisation system of  $\mathcal{C}$ .

We say that  $\mathcal{B} \hookrightarrow \mathcal{A}$  is **equationally definable** if there exists a set of equations  $\Phi$  in  $\mathcal{C}$ , such that an object  $A$  of  $\mathcal{A}$  lies in  $\mathcal{B}$  iff  $IA \models \Phi$  (where  $\Phi$  and  $\models$  are as in (6)). We say that  $\mathcal{B}$  is **closed under HSP** if and only if

- 1 For all morphisms  $e : B \rightarrow B'$  such that  $e \in E$  and  $Ie$  is a quotient, we have that  $B \in \mathcal{B}$  implies  $B' \in \mathcal{B}$ .
- 2 For all morphisms  $m : B \rightarrow B'$  such that  $m \in M$  we have  $B' \in \mathcal{B}$  implies  $B \in \mathcal{B}$ .
- 3 If  $B_i$  are in  $\mathcal{B}$  then their product in  $\mathcal{A}$  is an object of  $\mathcal{B}$ .

**Remark 3.3.** In general, the inclusion functor  $I$  does not preserve epimorphisms. We will assume that the arrows in  $M$  are monomorphisms. Being a right adjoint,  $I$  preserves products and monomorphisms, but we cannot infer from  $B' \rightarrow IB$  being a monomorphism in  $\mathcal{C}$  that  $B'$  is (isomorphic to an object) in  $\mathcal{A}$ .

**Remark 3.4.** The third item of Definition 3.2 makes sense only if  $\mathcal{A}$  has products. But  $\mathcal{A}$  is complete, since  $\mathcal{A}$  is a full reflective category of a complete category, see (Borceux, 1994, Proposition 3.5.3).

If  $\mathcal{C}$  is a category monadic over  $\mathbf{Set}^{\kappa}$  for some cardinal  $\kappa$  and  $\mathcal{A}$  is a full reflective subcategory of  $\mathcal{C}$ , then  $\mathcal{A}$  is complete and is well-powered because  $\mathcal{C}$  is. Hence we can equip  $\mathcal{A}$  with a strong-epi/mono factorisation system (Borceux, 1994, 4.4.3). We can prove:

**Theorem 3.5.** Let  $\mathcal{C}$  be a category monadic over  $\mathbf{Set}^\kappa$  for some cardinal  $\kappa$  and  $\mathcal{A}$  a full reflective subcategory of  $\mathcal{C}$ , such that the left adjoint of the inclusion functor preserves monomorphisms. Then  $\mathcal{B} \subseteq \mathcal{A}$  is closed under HSP in the sense of Definition 3.2 if and only if  $\mathcal{B}$  is equationally definable.

*Proof.* We will use  $U$  to denote the forgetful functor  $\mathcal{C} \rightarrow \mathbf{Set}^\kappa$  and  $F$  to denote its left adjoint.  $I$  denotes the full and faithful functor  $\mathcal{A} \rightarrow \mathcal{C}$  and  $R$  denotes its left adjoint.

Note that in  $\mathcal{A}$  (as in  $\mathcal{C}$ ) strong epis coincide with extremal epis (Borceux, 1994, 4.3.7) and with regular epis (Adámek et al., 1990, 14.14, 14.22). The proof of the theorem relies on the following two properties

$$e \text{ regular epi in } \mathcal{C} \Rightarrow Re \text{ regular epi in } \mathcal{A} \quad (7)$$

$$m \text{ mono in } \mathcal{A} \Rightarrow Im \text{ mono in } \mathcal{C} \quad (8)$$

(7) holds because  $R$  is a left-adjoint and (8) because  $I$  is a right adjoint. Also note that we have the converse of (8), since  $I$  is full and faithful.

Let us prove that equational definability implies closure under HSP. Let  $\mathcal{B}$  be an equationally definable subcategory of  $\mathcal{A}$ . That means that there exists an equationally definable subcategory  $\mathcal{B}'$  of  $\mathcal{C}$  such that  $B$  is an object of  $\mathcal{B}$  iff  $IB$  is an object of  $\mathcal{B}'$ . The proof of the fact that  $\mathcal{B}$  is closed under HSP in the sense of Definition 3.2, follows from the HSP theorem applied for  $\mathcal{B}'$  and the following observations:

- 1 The quotients  $e \in E$  considered in Definition 3.2 are exactly those for which  $Ie$  is a regular epimorphism in  $\mathcal{C}$ .
- 2  $I$  preserves monomorphisms.
- 3  $I$  preserves products.

Conversely, let  $\mathcal{B}$  be a subcategory of  $\mathcal{A}$ , closed under HSP, as in the previous definition. We will prove that  $\mathcal{B}$  is equationally definable. We proceed in three steps:

*Step 1 (construction of the equations that define  $\mathcal{B}$ ):* Let  $C$  be an arbitrary object of  $\mathcal{C}$ . We will consider all morphisms  $f_i : RC \rightarrow B_i$  in  $\mathcal{A}$  such that  $B_i$  is in  $\mathcal{B}$ . For each  $i$ , the corresponding morphism in  $\mathcal{C}$ ,  $f_i^\sharp : C \rightarrow IB_i$  factors in  $\mathcal{C}$ :

$$\begin{array}{ccc} C & & \\ & \searrow^{e_i} & \\ & & \overline{B_i} \\ & \swarrow_{m_i} & \\ & & IB_i \\ & \uparrow_{f_i^\sharp} & \end{array}$$

We will denote by  $\eta$  and  $\epsilon$  the unit, respectively co-unit, of the adjunction  $R \dashv I$ . One can easily show that the following diagram commutes:

$$\begin{array}{ccc}
 RC & & \\
 \downarrow f_i & \searrow Re_i & \\
 & & R\overline{B}_i \\
 & \swarrow \epsilon_{B_i} \circ Rm_i & \\
 B_i & & 
 \end{array} \quad (9)$$

Since  $R$  preserves regular epis and monos, and  $\epsilon_{B_i}$  is an isomorphism ( $I$  is full and faithful), we have that (9) is a factorisation of  $f_i$  in  $\mathcal{A}$ . But  $\mathcal{B}$  is closed under subobjects, hence  $R\overline{B}_i$  is actually an object of  $\mathcal{B}$ . Since  $\mathcal{C}$  is co-well-powered, there is only a proper set of different  $e_i$  up to isomorphism, so we can take the product  $P$  of the objects of the form  $R\overline{B}_i$ , obtained as above.  $P$  is again an object of  $\mathcal{B}$ , and we have a morphism  $\alpha : RC \rightarrow P$ , uniquely determined by the  $Re_i$ . We consider a factorisation in  $\mathcal{C}$  of the adjoint map  $\alpha^\sharp : C \rightarrow IP$ :

$$\begin{array}{ccc}
 C & & \\
 \downarrow \alpha^\sharp & \searrow e & \\
 & & Q_C \\
 & \swarrow m & \\
 IP & & 
 \end{array} \quad (10)$$

Using a similar argument to the above we deduce that  $RQ_C$  is an object of  $\mathcal{B}$  and the following diagram commutes:

$$\begin{array}{ccc}
 RC & \xrightarrow{Re} & RQ_C \\
 \downarrow f_i & & \downarrow \epsilon_P \circ Rm \\
 B_i & \xleftarrow{\epsilon_B \circ Rm_i} R\overline{B}_i \xleftarrow{\pi_i} & P
 \end{array} \quad (11)$$

We consider the class of equations  $\mathcal{E}$  of the form  $FX \rightarrow Q_{FX}$  for all sets  $X$ , and denote by  $\mathcal{B}'$  the subcategory of  $\mathcal{C}$  defined by these equations.

*Step 2 ( $\mathcal{B}$  is contained in the class defined by the equations  $\mathcal{E}$ ):* We show that if an object  $B$  of  $\mathcal{A}$  lies in  $\mathcal{B}$ , then  $IB$  satisfies the equations in  $\mathcal{E}$ . Let  $B$  be an object of  $\mathcal{B}$ , and let  $u : FX \rightarrow IB$  be an arbitrary morphism. For the adjoint morphism  $u_\sharp : RFX \rightarrow B$ , one can construct a morphism  $g : RQ_{FX} \rightarrow B$ , obtained as in diagram (11), such that  $g \circ Re = u_\sharp$ .

It is easy to see that  $g^\sharp : Q_{FX} \rightarrow IB$  makes the following diagram commutative. This shows that  $IB$  satisfies the equation  $e : FX \rightarrow Q_{FX}$ :

$$\begin{array}{ccc}
 FX & \xrightarrow{e} & Q_{FX} \\
 \downarrow u & & \swarrow g^\sharp \\
 IB & & 
 \end{array}
 \tag{12}$$

*Step 3 (The subcategory defined by the equations  $\mathcal{E}$  is contained in  $\mathcal{B}$ ):* Let  $B$  an object in  $\mathcal{B}$  such that  $IB$  satisfies the equations in  $\mathcal{E}$ . In particular  $IB$  satisfies  $FUIB \rightarrow Q_{FUIB}$ , so there exists  $v : Q_{FUIB} \rightarrow IB$  such that  $v \circ e = \epsilon'_{IB}$ , where  $\epsilon'$  is the counit of the adjunction  $F \dashv U$ . Since  $\epsilon'$  is a regular epi, then  $v$  is also a regular epi. We have that the composition  $\epsilon_B \circ Rv : RQ_{FUIB} \rightarrow B$  is a regular epi in  $\mathcal{A}$ . Since the codomain of  $v$  is in the image of  $I$ ,  $IRv$  is also a regular epi, therefore so is  $I(\epsilon_B \circ Rv)$ . Using the fact that  $\mathcal{B}$  is closed under H, and that  $RQ_{FUIB}$  is already in  $\mathcal{B}$ , we can conclude that  $B \in \mathcal{B}$ .  $\square$

The next proposition allows us to lift an adjunction between two categories to an adjunction between categories of algebras for functors satisfying some additional conditions.

**Proposition 3.6.** Let  $\langle R, I, \eta, \epsilon \rangle : \mathcal{A} \rightarrow \mathcal{B}$  be an adjunction. Let  $K$  and  $L$  be endofunctors on  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Suppose there exist natural transformations:  $\kappa : RK \rightarrow LR$  and  $\lambda : LI \rightarrow IK$  making the following diagrams commute:

$$\begin{array}{ccc}
 LIR & \xrightarrow{\lambda R} & IKR & \xrightarrow{I\kappa} & IRL \\
 & \swarrow L\eta & & & \searrow \eta L \\
 & & L & & \\
 & & (C_1) & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 KRI & \xrightarrow{\kappa I} & RLI & \xrightarrow{R\lambda} & RIK \\
 & \swarrow K\epsilon & & & \searrow \epsilon K \\
 & & K & & \\
 & & (C_2) & & 
 \end{array}
 \tag{13}$$

Then there exists an adjunction  $\langle \bar{R}, \bar{I}, \bar{\eta}, \bar{\epsilon} \rangle : \text{Alg}(K) \rightarrow \text{Alg}(L)$ , such that  $U_{\mathcal{A}}\bar{R} = RU_{\mathcal{B}}$  and  $IU_{\mathcal{A}} = U_{\mathcal{B}}\bar{I}$ , where  $U_{\mathcal{A}}$  and  $U_{\mathcal{B}}$  denote the forgetful functors as in the next diagram:

$$\begin{array}{ccc}
 \text{Alg}(K) & \begin{array}{c} \xleftarrow{\bar{R}} \\ \perp \\ \xrightarrow{\bar{I}} \end{array} & \text{Alg}(L) \\
 \downarrow U_{\mathcal{A}} & & \downarrow U_{\mathcal{B}} \\
 K \circlearrowleft \mathcal{A} & \begin{array}{c} \xleftarrow{R} \\ \perp \\ \xrightarrow{I} \end{array} & \mathcal{B} \circlearrowright L
 \end{array}
 \tag{14}$$

*Proof.* First let us define the functor  $\bar{I}$ . Let  $f : KA \rightarrow A$  be a  $K$ -algebra. We define  $\bar{I}(A, f) := (IA, If \circ \lambda_A)$ . For an arbitrary morphism of  $K$ -algebras  $u : (A, f) \rightarrow (A', f')$ , we define  $\bar{I}(u) = Iu$ . The fact that  $Iu$  is a morphism of  $L$ -algebras follows from the

commutativity of the outer square of the next diagram:

$$\begin{array}{ccccc}
 LIA & \xrightarrow{\lambda_A} & IKA & \xrightarrow{If} & IA \\
 \downarrow LIu & & \downarrow IKu & & \downarrow Iu \\
 LIA' & \xrightarrow{\lambda'_A} & IKA' & \xrightarrow{If'} & IA'
 \end{array} \quad (15)$$

But the small squares commute, the former because  $\lambda$  is a natural transformation, and the latter because  $u$  is a  $K$ -algebra morphism. It is obvious that  $\bar{I}$  is a functor and that  $IU_A = U_B\bar{I}$ . The functor  $\bar{R}$  is defined similarly: if  $g : LB \rightarrow B$  is a  $L$ -algebra, we define  $\bar{R}(B, g) = (RB, Rg \circ \kappa_B)$ . If  $v : (B, g) \rightarrow (B', g')$  is a  $L$ -algebra morphism, we define  $\bar{R}(v) = Rv$ . The fact that  $Rv$  is indeed a  $K$ -algebra morphism is verified easily, using the naturality of  $\kappa$  and the fact that  $v$  is a  $L$ -algebra morphism.

In order to prove that  $\bar{R}$  is left adjoint to  $\bar{I}$ , we will show that the unit  $\eta$  and the counit  $\epsilon$  of the adjunction  $R \dashv I$  are  $L$ -algebra and  $K$ -algebra morphisms respectively. This follows from the hypothesis (13) and the naturality of  $\eta$  and  $\epsilon$  respectively.

Once this is achieved,  $\eta$  can be lifted to a natural transformations  $\bar{\eta} : \text{id} \rightarrow \bar{I}\bar{R}$ , and similarly  $\epsilon$  can be lifted to a natural transformation  $\bar{\epsilon} : \bar{R}\bar{I} \rightarrow \text{id}$ . But  $\eta$  and  $\epsilon$  are the unit and the counit of the adjunction  $R \dashv I$ , therefore they satisfy the usual triangular equalities. Therefore,  $\bar{\eta}$  and  $\bar{\epsilon}$  satisfy the triangular equalities for  $\bar{R}$  and  $\bar{I}$ .  $\square$

**Remark 3.7.** The proposition has some useful special cases, under the additional assumption that  $I$  is full and faithful. For each of them, the commutativity of the diagrams (13) is straightforward to verify, using that the counit  $\epsilon$  is iso.

- 1 Suppose  $L$  is given and we want to find an appropriate  $K$ . Then it follows from the theorem that we can do this, provided there is a natural transformation  $\alpha : LIR \rightarrow IRL$  such that the diagram below commutes:

$$\begin{array}{ccc}
 LIR & \xrightarrow{\alpha} & IRL \\
 & \swarrow L\eta & \nearrow \eta L \\
 & L & 
 \end{array} \quad (16)$$

If this is the case, one defines  $K = RLI$ ,  $\kappa : KR = RLIR \rightarrow RL$  as the composition  $\epsilon_{RL} \circ R\alpha$  and  $\lambda : LI \rightarrow IRLI = IK$  as  $\eta_{LI}$ . Moreover, we have that  $LIR \rightarrow IRLIR$  is iso and that  $R\alpha$  and hence  $KR \rightarrow RL$  are iso. (If we don't find such an  $\alpha$  we might think of replacing  $L$  by  $IRLIR$  since we always have  $(IRLIR)IR \rightarrow IR(IRLIR)$ ; we still have  $K \cong RLI$ .)

- 2 More generally, suppose we have given an iso  $\kappa : KR \rightarrow RL$ . Then we define  $\lambda = IK\epsilon \circ I\kappa^{-1}I \circ \eta LI$ . (Given  $K$  we can always find such a  $\kappa$ : Let  $L = IKR$  and  $\kappa = (\epsilon KR)^{-1} : KR \cong RIKR$ .)

Let  $\Sigma$  be a polynomial functor  $\Sigma X = \coprod_{i \in J} X^{n_i}$ . If both  $\mathcal{A}$  and  $\mathcal{B}$  have (co)products, then  $\Sigma$  is defined on both categories, so it makes sense to write  $R\Sigma \cong \Sigma R$ . The following

corollary says that for polynomial functors  $\Sigma$  the adjunction always lifts from the base categories to the categories of  $\Sigma$ -algebras.

**Corollary 3.8.** Let  $\langle R, I, \eta, \epsilon \rangle : \mathcal{A} \dashv \mathcal{B}$  be an adjunction such that  $I$  is full and faithful. Further assume that both categories have coproducts and finite products and that  $R$  preserves finite products. Consider an endofunctor  $\Sigma X = \coprod_{i \in J} X^{n_i}$  on  $\mathcal{A}$  and on  $\mathcal{B}$ . Then the adjunction lifts to an adjunction  $\langle \bar{R}, \bar{I}, \eta, \epsilon \rangle : \text{Alg}(\Sigma) \dashv \text{Alg}(\Sigma)$ .

*Proof.* We use item 2 of the remark above and calculate  $R\Sigma A = R(\coprod_{i \in J} A^{n_i}) \cong \coprod_{i \in J} (RA)^{n_i} = \Sigma RA$ .  $\square$

**Remark 3.9.**  $\bar{R}$  preserves finite limits whenever  $R$  does.

*Proof.* Assume  $(B, g) = \varprojlim (B_i, g_i)$  is a finite limit in  $\text{Alg}(L)$ . Since  $U_{\mathcal{B}}$  preserves all limits, we have that  $B = \varprojlim (B_i)$  in  $\mathcal{B}$ , and therefore  $RB = \varprojlim (RB_i)$ . Denote  $\bar{R}(B_i, g_i)$  by  $(RB_i, f_i)$  and by  $\pi_i : RB \rightarrow RB_i$  the morphisms of the limiting cone. For each index  $i$  we have a map  $p_i : KRB \rightarrow RB_i$  obtained as the composition  $f_i \circ K\pi_i$ . From the universal property, we obtain a map  $f : KRB \rightarrow RB$ , such that each  $\pi_i$  is a  $K$ -algebra morphism from  $(RB, f)$  to  $(RB_i, f_i)$ . We prove next that  $(RB, f) = \varprojlim (RB_i, f_i)$  in  $\text{Alg}(K)$ . Assume that we have a cone  $q_i : (C, h) \rightarrow (RB_i, f_i)$ . Since  $RB$  is a limit in  $\mathcal{A}$  we get a unique map  $k : C \rightarrow RB$ , such that  $\pi_i \circ k = q_i$ . We need to show that  $k$  is a  $K$ -algebra morphism. To this end we will use the uniqueness of a morphism from  $KC \rightarrow RB$  that makes the relevant diagrams commutative.  $\square$

### 3.2. HSP theorem for nominal sets and sheaf algebras

In this section we will prove an HSP theorem for algebras over  $\text{Sh}(\mathbb{I}^{op})$ . We will call these algebras ‘sheaf algebras’. Some of them, given by particular signatures, correspond, in a sense that will be made precise in Section 6.1, to nominal algebras, (Gabbay and Mathijssen, 2009). The signature will be given by a functor  $L$  on  $\text{Set}^{\mathbb{I}}$  that has a finitary presentation. On  $\text{Sh}(\mathbb{I}^{op})$  we can define the functor  $\tilde{L}$  as  $RLL$ . Throughout this section we will use the notations from Diagram (2). The goal of this section is to derive an HSP theorem for  $\text{Alg}(\tilde{L})$  from Theorem 3.5. To this end we will need to impose some reasonable conditions on the functor  $L$ .

**Theorem 3.10. HSP theorem for ‘nominal algebras’.** In the situation of Diagram (2), let  $L$  be an endofunctor with a finitary presentation on  $\text{Set}^{\mathbb{I}}$  and let  $\tilde{L}$  be such that  $\tilde{L}R \cong RL$ . Then a full subcategory of  $\text{Alg}(\tilde{L})$  is closed under HSP if and only if it is equationally definable.

*Proof.* By Theorem 2.6, we have that  $\text{Alg}(L)$  is monadic over  $\text{Set}^{\mathcal{P}_f(\mathcal{N})}$ , so it has a regular factorisation system. By Proposition 3.6, we can lift the adjunction  $R \dashv I$  of Proposition 2.1 to an adjunction  $\bar{R} \dashv \bar{I}$  between the categories of  $\tilde{L}$ -algebras and  $L$ -algebras. Since  $\bar{R}$  preserves finite limits, it preserves monomorphisms, so we can apply Theorem 3.5.  $\square$

**Remark 3.11.** Notice that in the case of algebras over  $\mathbf{Set}^{\mathbb{I}}$ , we can assume that the equations defining a subcategory closed under HSP involve only finitely many variables. It is enough to prove that if  $\mathcal{A} \subseteq \mathbf{Alg}(L)$  is closed under HSP then  $\mathcal{A}$  is also closed under directed unions, see also Remark 3.1. The idea of the proof is to construct the directed union of a directed family  $(X_i)_{i \in J}$  of algebras in  $\mathcal{A}$ , as a homomorphic image of a subalgebra of a product of algebras  $X_i$ , as in the proof of (Adámek and Rosický, 1994, Theorem 3.9). The subtlety here is that the product considered there may be empty in the general many-sorted case, even if some of the  $X_i$  are not. However, in the case of algebras over  $\mathbf{Set}^{\mathbb{I}}$  we can see that if one algebra  $X_{i_0}$  has the underlying presheaf non-empty, say, for example  $X_{i_0}(S) \neq \emptyset$ , then for all  $j \geq i_0$  and for all sets  $T$  of cardinality larger than that of  $S$  we also have that  $X_j(T)$  is non-empty. We can consider the product of the  $X_j$  for  $j \geq i_0$ , and this is non-empty.

#### 4. Concrete syntax

This section illustrates the concrete syntax obtained from the abstract category theoretic treatment of Section 3. In the first part of this section we give a presentation for the category  $\mathbf{Set}^{\mathbb{I}}$ . Then we define a ‘shift’ functor  $\delta$  on  $\mathbf{Set}^{\mathbb{I}}$ , which corresponds to the abstraction operator of (Gabbay and Pitts, 1999) and to the ‘shift’ functor on  $\mathbf{Set}^{\mathbb{F}}$  from (Fiore et al., 1999). Next, we illustrate the concrete syntax obtained in our setting, by giving a theory for the  $\lambda$ -calculus. We consider an endofunctor  $L$  on  $\mathbf{Set}^{\mathbb{I}}$  given by

$$LX = \mathcal{N} + \delta X + X \times X \tag{17}$$

where  $\mathcal{N}$  denotes the inclusion functor  $\mathcal{N} : \mathbb{I} \rightarrow \mathbf{Set}$ . In order to show that the HSP theorem holds for  $\tilde{L}$ -algebras, we need to prove that  $L$  satisfies the conditions in Theorem 3.10. This is actually a particular case of Proposition 5.4 that will be proved in the next section. In Section 4.2 we prove that the sheaf of  $\lambda$ -terms up to  $\alpha$ -equivalence is the initial algebra for  $\tilde{L} = RLI$  and we give the equations that characterise the subalgebra of  $\lambda$ -terms modulo  $\alpha\beta\eta$ -equivalence.

##### 4.1. A presentation for $\mathbf{Set}^{\mathbb{I}}$ and $\delta$

In order to spell out some equations in our setting we need a presentation for  $\mathbf{Set}^{\mathbb{I}}$ . The operation symbols should correspond to morphisms that generate all the arrows in  $\mathbb{I}$ . One would be tempted to use operation symbols of the form  $(a, b)_S$ , which correspond to swapping the names  $a$  and  $b$  of a set  $S$ , and  $w_{S,a}$ , which correspond to inclusions of  $S$  into  $S \cup \{a\}$ . However swappings and inclusions fail to generate all the bijections in  $\mathbb{I}$ . For example, if  $a \neq b$  and  $a, b \notin S$ , they cannot generate a bijection from  $S \cup \{a\}$  to  $S \cup \{b\}$  which maps  $a$  to  $b$  and acts as identity on the remaining elements of  $S$ .

This example suggests the following set  $\Sigma_{\mathbb{I}}$  of operation symbols with specified arity:

$$\begin{aligned} (b/a)_S : S \cup \{a\} &\rightarrow S \cup \{b\} & a \neq b, a \notin S, b \notin S \\ w_{S,a} : S &\rightarrow S \cup \{a\} & a \notin S \end{aligned} \tag{18}$$

We will refer to operation symbols of the form  $(b/a)_S$  as ‘substitutions’ and to operations symbols of the form  $w_{S,a}$  as ‘inclusions’. When the arity can be inferred from the context, or is irrelevant, we will omit  $S$  from the subscript.

We consider the set  $E_{\mathbb{I}}$  of equations of the form:

$$\begin{aligned}
(a/b)_S(b/a)_S(x) = x & & : S \cup \{a\} & \quad (E_1) \\
(b/a)_{S \cup \{a\}}(d/c)_{S \cup \{a\}}(x) = (d/c)_{S \cup \{b\}}(b/a)_{S \cup \{c\}}(x) & & : S \cup \{b, d\} & \quad (E_2) \\
(c/b)_S(b/a)_S(x) = (c/a)_S & & : S \cup \{c\}(x) & \quad (E_3) \\
(b/a)_{S \cup \{c\}}w_{S \cup \{a\},c}(x) = w_{S \cup \{b\},c}(b/a)_S & & : S \cup \{c, b\}(x) & \quad (E_4) \\
(b/a)_S w_{S,a}(x) = w_{S,b}(x) & & : S \cup \{b\} & \quad (E_5) \\
w_{S \cup \{b\},a}w_{S,b}(x) = w_{S \cup \{a\},b}w_{S,a}(x) & & : S \cup \{a, b\} & \quad (E_6)
\end{aligned} \tag{19}$$

**Theorem 4.1.**  $(\Sigma_{\mathbb{I}}, E_{\mathbb{I}})$  is a presentation for  $\mathbf{Set}^{\mathbb{I}}$ .

Next we will define a ‘shift’ functor  $\delta$  on  $\mathbf{Set}^{\mathbb{I}}$ . Assume  $P : \mathbb{I} \rightarrow \mathbf{Set}$  is a presheaf and  $S \subseteq \mathcal{N}$  is a finite set of names. We define an equivalence relation  $\equiv$  on  $\coprod_{a \notin S} P(S \cup \{a\})$ .

If  $a, b \notin S$ ,  $x \in P(S \cup \{a\})$  and  $y \in P(S \cup \{b\})$  we will say that  $x$  and  $y$  are equivalent if and only if  $P((b/a)_S)(x) = y$ . We define  $(\delta P)(S)$  as the set of equivalence classes of the elements of  $\coprod_{a \notin S} P(S \cup \{a\})$ . If  $x \in P(S \cup \{a\})$  the equivalence class of  $x$  is denoted by  $\bar{x}^{S,a}$ .

If  $f : S \rightarrow T$  is a morphism in  $\mathbb{I}$  and  $a \notin S \cup T$ ,  $f + a : S \cup \{a\} \rightarrow T \cup \{a\}$  denotes the function which restricted to  $S$  is  $f$  and which maps  $a$  to  $a$ . We define  $(\delta A)(f)(\bar{x}^{S,a}) = \overline{A(f+a)(x)}^{T,a}$  for some  $a \notin S$ . One can easily check that  $(\delta A)(f)$  is well defined and that  $\delta$  is a functor.

Next, we give a presentation for  $\delta$ . For each finite subset of names  $S \subseteq \mathcal{N}$  and for each  $a \notin S$  we consider an operation symbol  $[a]_S : S \cup \{a\} \rightarrow S$ , and we will denote by  $\Sigma_{\delta}$  the corresponding functor on  $\mathbf{Set}^{\mathcal{P}_f(\mathcal{N})}$ . This is given by  $(\Sigma_{\delta}X)_S = \coprod_{a \notin S} \{[a]_S\} \times X_{S \cup \{a\}}$ . We

denote by  $U : \mathbf{Set}^{\mathbb{I}} \rightarrow \mathbf{Set}^{\mathcal{P}_f(\mathcal{N})}$  the forgetful functor and by  $F$  its left adjoint. For any functor  $P : \mathbb{I} \rightarrow \mathbf{Set}$  we can give an interpretation of these operation symbols, captured by a natural transformation  $\rho_P : \Sigma_{\delta}UP \rightarrow U\delta P$  defined by:

$$\forall \alpha \in P(S \cup \{a\}) \quad ([a]_S, \alpha) \mapsto \bar{\alpha}^{S,a} \in (U\delta P)(S)$$

The equations should correspond to the kernel pair of the adjoint transpose  $\rho_P^{\sharp} : F\Sigma_{\delta}UP \rightarrow \delta P$ , as in Definition 2.5. We will use the fact that for any  $X = (X_S)_{S \in \mathcal{P}_f(\mathcal{N})}$  we have  $(FX)_S = \coprod_{T \in \mathcal{P}_f(\mathcal{N})} X_T \cdot \text{hom}(T, S)$ , where  $\cdot$  is the copower. For  $f : T \rightarrow S$  and  $x \in X_T$

we denote by  $fx$  the element of  $(FX)_S$  which is the copy of  $f$  corresponding to  $x$ . The equations  $E_{\delta}$  will have the form:

$$\begin{aligned}
(c/b)_S[a]_{S \cup \{b\}}t &= [a]_{S \cup \{c\}}(c/b)_{S \cup \{a\}}t & t : S \cup \{a, b\} \\
[a]_S t &= [b]_S (b/a)_S t & t : S \cup \{a\} \\
w_{S,b}[a]_S t &= [a]_{S \cup \{b\}}w_{S \cup \{a\},b}t & t : S \cup \{a\}
\end{aligned} \tag{20}$$

**Theorem 4.2.**  $(\Sigma_\delta, E_\delta)$  is a presentation for  $\delta$ .

**Notation 4.3.** We will denote the element  $\bar{\alpha}^{S,a} \in (\delta P)(S)$  by  $\{[a]_{S\alpha}\}_{\delta P}$ .

#### 4.2. Axioms for the $\lambda$ -calculus

The  $\alpha$ -equivalence classes of  $\lambda$ -terms over  $\mathcal{N}$  form a sheaf  $\Lambda_\alpha$  in  $\text{Sh}(\mathbb{I}^{op})$ . Indeed, we can define  $\Lambda_\alpha(S)$  as the set of  $\alpha$ -equivalence classes of  $\lambda$ -terms with free variables in  $S$ . On functions  $\Lambda_\alpha$  acts by renaming the free variables.

We consider the endofunctor  $L$  on  $\text{Set}^{\mathbb{I}}$ , defined by (17) and the endofunctor  $\tilde{L}$  on  $\text{Sh}(\mathbb{I}^{op})$  defined as  $RLI$ . In a similar fashion as in (Fiore et al., 1999), we will show that  $\Lambda_\alpha$  is isomorphic to the initial algebra  $\mathcal{I}_{\tilde{L}}$  for  $\tilde{L}$ .

First let us notice that the underlying presheaf of  $\mathcal{I}_{\tilde{L}}$  is the initial algebra  $\mathcal{I}_L$  for  $L$ . Indeed, one can prove

**Lemma 4.4.** We have  $I\mathcal{I}_{\tilde{L}} = \mathcal{I}_L$ .

*Proof.* We can check that  $\tilde{L}$  preserves  $\omega$ -chains, so the initial algebra  $\mathcal{I}_{\tilde{L}}$  is computed as the colimit of the sequence

$$\tilde{0} \rightarrow \tilde{L}\tilde{0} \rightarrow \tilde{L}^2\tilde{0} \rightarrow \cdots \rightarrow \mathcal{I}_{\tilde{L}} \quad (21)$$

where  $\tilde{0}$  is just the empty sheaf. Let us denote by  $0$  the empty presheaf. Similarly  $\mathcal{I}_L$  is the colimit of the initial sequence for  $L$ :

$$0 \rightarrow L0 \rightarrow L^20 \rightarrow \cdots \rightarrow \mathcal{I}_L. \quad (22)$$

Using the observation that  $L$  preserves sheaves and the fact that  $I\tilde{0} = 0$  we can easily verify that  $I\tilde{L}^n\tilde{0} \simeq L^n0$  for all natural numbers  $n$ . But  $I$  preserves filtered colimits, so we have that  $I\mathcal{I}_{\tilde{L}} \simeq \mathcal{I}_L$ .  $\square$

We consider a functor  $\Sigma : \text{Set}^{\mathbb{I}} \rightarrow \text{Set}^{\mathbb{I}}$ , defined by

$$\Sigma X = \mathcal{N} + \mathcal{N} \times X + X \times X. \quad (23)$$

Notice that  $\Sigma$  preserves sheaves and that the initial algebra for  $\Sigma$ , let us denote it by  $\mathcal{I}_\Sigma$ , is just the presheaf of all  $\lambda$ -terms. Using a similar argument as above we can see that  $I\mathcal{I}_{\tilde{\Sigma}} = \mathcal{I}_\Sigma$ .  $\mathcal{I}_{\tilde{\Sigma}}$  is the sheaf of all  $\lambda$ -terms. We will prove the isomorphism between  $\Lambda_\alpha$  and  $\mathcal{I}_{\tilde{L}}$  by constructing an epimorphism  $\mathcal{I}_{\tilde{\Sigma}} \rightarrow \mathcal{I}_{\tilde{L}}$  in  $\text{Sh}(\mathbb{I}^{op})$  that identifies exactly  $\alpha$ -equivalent terms.

First let us prove the next lemma:

**Lemma 4.5.** There exists a natural transformation  $\theta : \mathcal{N} \times - \rightarrow \delta$  such that for any presheaf  $X \in \text{Sh}(\mathbb{I}^{op})$  and any finite set of names  $S \subseteq \mathcal{N}$  we have that  $\theta(a, x) = \theta(b, y)$  for some  $a, b \in S$  and  $x, y \in X(S)$  iff  $X(\sigma_{a,b})(x) = y$ . Moreover if  $X$  is a sheaf, then  $\theta_X$  is a sheaf epimorphism.

*Proof.* We define  $\theta_X(S) : \mathcal{N}(S) \times X(S) \rightarrow (\delta X)(S)$  by

$$(a, x) \mapsto (\delta X)(w_{S \setminus \{a\}, a})(\{[a]_{S \setminus \{a\}} x\}_{\delta X}) \quad (24)$$

where  $w_{S \setminus \{a\}, a}$  is the inclusion of  $S \setminus \{a\}$  into  $S$ . It is not difficult to check that this is indeed a natural transformation. Assume now that  $(a, x), (b, y) \in \mathcal{N}(S) \times X(S)$  and  $c \in \mathcal{N} \setminus S$ . We have that  $\theta(a, x) = \theta(b, x)$  is equivalent to

$$(\delta X)(w_{S \setminus \{a\}, a})(\{[a]_{S \setminus \{a\}} x\}_{\delta X}) = (\delta X)(w_{S \setminus \{b\}, b})(\{[b]_{S \setminus \{b\}} y\}_{\delta X}) \quad (25)$$

But

$$\begin{aligned} (\delta X)(w_{S \setminus \{a\}, a})(\{[a]_{S \setminus \{a\}} x\}_{\delta X}) &= (\delta X)(w_{S \setminus \{a\}, a})(\{[c]_{S \setminus \{a\}} (c/a)_{S \setminus \{a\}} x\}_{\delta X}) \\ &= \{[c]_S X(w_{S \setminus \{a\}, a} + c)((c/a)_{S \setminus \{a\}}(x))\}_{\delta X} \\ &= \{[c]_S X(w_{S \setminus \{a\} \cup \{c\}, a}((c/a)_{S \setminus \{a\}}(x)))\}_{\delta X} \end{aligned} \quad (26)$$

Similarly,  $(\delta X)(w_{S \setminus \{b\}, b})(\{[b]_{S \setminus \{b\}} y\}_{\delta X}) = \{[c]_S X(w_{S \setminus \{b\} \cup \{c\}, b}((c/b)_{S \setminus \{b\}}(y)))\}_{\delta X}$

Therefore (25) is equivalent to

$$\{[c]_S X(w_{S \setminus \{a\} \cup \{c\}, a}((c/a)_{S \setminus \{a\}}(x)))\}_{\delta X} = \{[c]_S X(w_{S \setminus \{b\} \cup \{c\}, b}((c/b)_{S \setminus \{b\}}(y)))\}_{\delta X}$$

Using that  $w_{S \setminus \{a\} \cup \{c\}, a} = w_{S \setminus \{b\} \cup \{c\}, b}(a/b)_{S \setminus \{a, b\} \cup \{c\}}$ , we can derive that (25) is equivalent to

$$X(w_{S \setminus \{b\} \cup \{c\}, b} X((a/b)_{S \setminus \{a, b\} \cup \{c\}}(c/a)_{S \setminus \{a\}}(x))) = X(w_{S \setminus \{b\} \cup \{c\}, b} X((c/b)_{S \setminus \{b\}}(y))).$$

$X$  is a sheaf, and therefore it preserves monomorphisms, hence we have that

$$X((a/b)_{S \setminus \{a, b\} \cup \{c\}}(c/a)_{S \setminus \{a\}}(x)) = X((c/b)_{S \setminus \{b\}}(y))$$

or equivalently,

$$X((b/c)_{S \setminus \{b\}}(a/b)_{S \setminus \{a, b\} \cup \{c\}}(c/a)_{S \setminus \{a\}}(x)) = y$$

which means that  $X(\sigma_{a,b})(x) = y$ .

In order to prove the last statement of the lemma, we use the characterisation of sheaf epimorphisms given in Proposition 2.3. Let  $\{[c]_S y\}_{\delta X}$  be an arbitrary element of  $(\delta X)(S)$ . We have that  $c \notin S$  and  $y \in X(S \cup \{c\})$ . The conclusion follows from the fact that  $\theta_X(S \cup \{c\})(c, y) = (\delta X)(w_{S, c})(\{[c]_S y\}_{\delta X})$ .  $\square$

**Proposition 4.6.** The sheaf of  $\alpha$ -equivalence classes of  $\lambda$ -terms is isomorphic to the initial  $\tilde{L}$ -algebra  $\mathcal{I}_{\tilde{L}}$ .

*Proof.* This can be obtained by an inductive argument on the structure of the  $\lambda$ -terms. Using the natural transformation  $\theta$  defined above we can construct a natural transformation  $\vartheta : \Sigma \rightarrow L$  defined as  $\vartheta_X = \text{id}_{\mathcal{N}} + \theta + \text{id}_X \times \text{id}_X$ . Now we can define inductively a natural transformation  $\zeta^{(n)} : \Sigma^n \rightarrow L^n$ . Explicitly,  $\zeta^{(0)} = \text{id}_0$  and  $\zeta_X^{(n+1)} =$

$L^n(\vartheta_X)\zeta_{\Sigma X}^{(n)}$ . We have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma 0 & \longrightarrow & \Sigma^2 0 & \longrightarrow & \cdots \longrightarrow \mathcal{I}_\Sigma \\
 \downarrow \zeta^{(0)} & & \downarrow \zeta^{(1)} & & \downarrow \zeta^{(2)} & & \downarrow \zeta \\
 0 & \longrightarrow & L0 & \longrightarrow & L^2 0 & \longrightarrow & \cdots \longrightarrow \mathcal{I}_L
 \end{array} \tag{27}$$

where  $\zeta$  is obtained by taking the colimit. As seen above,  $\mathcal{I}_L$  and  $\mathcal{I}_\Sigma$  are the underlying presheaves for the sheaves  $\mathcal{I}_{\tilde{L}}$  and  $\mathcal{I}_{\tilde{\Sigma}}$ , respectively.

Using Lemma 4.5, we can argue inductively that  $\zeta_X^{(n)}$  is a sheaf epimorphism for all  $n$  and for all sheaves  $X$ . One can verify that this implies that  $\zeta$  is a sheaf epimorphism. If two terms in  $\mathcal{I}_\Sigma$  are identified by  $\zeta$ , then they must be identified at some stage  $n$  by  $\zeta^{(n)}$ . Using Lemma 4.5 again, we can show by induction that two terms are in the kernel of  $\zeta^{(n)}$  if and only if they are  $\alpha$ -equivalent.  $\square$

To illustrate the concrete syntax appearing in our setting, we give a presentation for the functor  $L$  and a theory over the signature given by  $L$  for  $\alpha\beta\eta$ -equivalence of  $\lambda$ -terms.

**Proposition 4.7.** The endofunctor  $L$  is presented by a set of operation symbols

$$\begin{aligned}
 a_S & : S \cup \{a\} \\
 \mathbf{app}_S & : S \times S \rightarrow S \\
 [a]_S & : S \cup \{a\} \rightarrow S
 \end{aligned}$$

where  $S$  is a finite set of names and  $a \notin S$ , and the following set of equations:

$$\begin{aligned}
 (b/a)_S a_S & = b_S & (\text{E}_0) \\
 w_{S \cup \{a\}, b} a_S & = a_{S \cup \{b\}} & (\text{E}_1) \\
 (c/b)_S ([a]_{S \cup \{b\}} t) & = \text{id}_{S \cup \{c\}}([a]_{S \cup \{c\}}(c/b)_{S \cup \{a\}} t) & t : S \cup \{a, b\} & (\text{E}_2) \\
 [a]_S t & = [b]_S (b/a)_S t & t : S \cup \{a\} & (\text{E}_3) \\
 w_{S, b} [a]_S t & = [a]_{S \cup \{b\}} w_{S \cup \{a\}, b} t & t : S \cup \{a\} & (\text{E}_4) \\
 w_{S, a} \mathbf{app}_S(t_1, t_2) & = \mathbf{app}_{S \cup \{a\}}(w_{S, a} t_1, w_{S, a} t_2) & t_1, t_2 : S & (\text{E}_5) \\
 (b/a)_S \mathbf{app}_{S \cup \{a\}}(t_1, t_2) & = \mathbf{app}_{S \cup \{b\}}((b/a)_S t_1, (b/a)_S t_2) & t_1, t_2 : S \cup \{a\} & (\text{E}_6)
 \end{aligned}$$

**Example 4.8.** The subalgebra of  $\text{Alg}(\tilde{L})$  of  $\lambda$ -terms modulo  $\alpha\beta\eta$ -equivalence is definable by the following equations, similar to (Clouston and Pitts, 2007, Fig. 4).

$$\begin{aligned}
 X : S; Y : S & \vdash \mathbf{app}_S([a]_S(w_a Y), X) = Y : S & (\beta-1) \\
 X : S & \vdash \mathbf{app}_S([a]_S(a_S), X) = X : S & (\beta-2) \\
 X : S \cup \{a, b\}; Y : S & \vdash \mathbf{app}_S([a]_S([b]_{S \cup \{a\}}(X)), Y) = \\
 & [b]_S(\mathbf{app}_{S \cup \{b\}}([a]_{S \cup \{b\}}(X), w_a Y)) : S & (\beta-3) \\
 X, Y : S \cup \{a\}; Z : S & \vdash \mathbf{app}_S([a]_S(\mathbf{app}_{S \cup \{a\}}(X, Y)), Z) = \\
 & \mathbf{app}_S(\mathbf{app}_S([a]_S(X), Z), \mathbf{app}_S([a]_S(Y), Z)) : S & (\beta-4) \\
 X : S \cup \{a\} & \vdash \mathbf{app}_{S \cup \{b\}}(w_b[a]_S(X), b_S) = (b/a)_S X : S \cup \{b\} & (\beta-5) \\
 X : S & \vdash [a]_S(\mathbf{app}_{S \cup \{a\}}(w_a X, a_S)) = X : S & (\eta)
 \end{aligned}$$

## 5. Uniform Theories

Gabbay (Gabbay, 2008) proves an HSP theorem for nominal algebras, or rather an HSPA-theorem: A class of nominal algebras is definable by a theory of nominal algebra iff it is closed under HSP and under abstraction.

Our equational logic is more expressive than Gabbay’s in the sense that more classes are equationally definable, namely all those closed under HSP where H refers not to closure under all quotients as in (Gabbay, 2008), but to the weaker property of closure under support-preserving quotients (i.e. quotients in the presheaf category). Of course, it is a question whether this additional expressivity is wanted. We therefore isolate a fragment of standard equational logic, which we call the *uniform fragment*, and define notions of uniform signature, uniform terms and uniform equations. The main idea is that a uniform equation  $t = u : T$ , for  $T$  a finite subset of  $\mathcal{N}$ , has an interpretation uniform in all sorts  $S$  containing  $T$ .

For this uniform fragment we are able to extend Theorem 3.11 to an HSPA theorem in the style of [9]: classes of sheaf algebras are definable by uniform equations if and only if they are closed under quotients, subalgebras, products and under abstraction.

We start with the observation that the theory of the  $\lambda$ -calculus up to  $\alpha\beta\eta$ -equivalence (Example 4.8) uses only particular operations: names (atoms in (Gabbay, 2008)), abstraction, and operations  $f_S : A^n(S) \rightarrow A(S)$  that are ‘uniform’ in  $S$ . This motivates us to consider sheaf algebras for signatures given by a particular class of functors, and specified by ‘uniform’ equations. Throughout this section we will assume that the endofunctor  $L$  on  $\mathbf{Set}^{\mathbb{I}}$  is presented by a set of operation symbols that form a presheaf in  $\mathbf{Set}^{\mathbb{I}}$ , say  $\mathcal{O}$ , and such that all the operation symbols in  $\mathcal{O}(T)$  have arity of the form  $T \times \cdots \times T \rightarrow T'$ , for some finite subset  $T' \subseteq T$ . We will use the notation

$$\mathbf{bind}(f) = T \setminus T' \quad (28)$$

Additionally we assume that if an operation symbol  $f_T \in \mathcal{O}(T)$  has arity  $T \times \cdots \times T \rightarrow T'$  and  $j : T \rightarrow S$  is an injective map, then  $\mathcal{O}(j)(f_T)$  has arity  $S \times \cdots \times S \rightarrow S \setminus j[\mathbf{bind}(f)]$ , where  $j[\mathbf{bind}(f)]$  denotes the direct image. For an injective map  $j : T \rightarrow S$  we write  $j \bullet f_T$  for  $\mathcal{O}(j)(f_T)$ . For simplicity, if  $j : T \rightarrow S$  is an inclusion we write simply  $f_S$  for  $j \bullet (f_T)$ . For example  $f_{T \cup \{a\}}$  stands for  $w_a \bullet f_T$ .

**Definition 5.1.** We will say that  $L$  has a *uniform presentation* if  $L$  is presented by operation symbols forming a presheaf as above, and a set of equations containing the following:

$$\begin{aligned} w_a f_T(x_1, \dots, x_n) &= f_{T \cup \{a\}}(w_a x_1, \dots, w_a x_n) \\ f_T(x_1, \dots, x_n) &= ((b/a)_{T \setminus \{a\}} \bullet f_T)((b/a)_{T \setminus \{a\}} x_1, \dots, (b/a)_{T \setminus \{a\}} x_n) \\ (b/a)_{T' \setminus \{a\}} f_T(x_1, \dots, x_n) &= ((b/a)_{T \setminus \{a\}} \bullet f_T)((b/a)_{T \setminus \{a\}} x_1, \dots, (b/a)_{T \setminus \{a\}} x_n) \end{aligned} \quad (29)$$

where in the second equation  $a \in \mathbf{bind}(f)$  and  $b \notin T$ , while in the third  $a \in T'$  and  $b \notin T$ .

Intuitively, these equations state that the operations are ‘equivariant’. If  $X$  is a presheaf, elements of  $LX(T')$  will be denoted by  $\{f_T(x_1, \dots, x_n)\}_X$ , where  $x_1, \dots, x_n \in X(T)$  and  $f_T$  has arity  $T \times \cdots \times T \rightarrow T'$ .

**Remark 5.2.** From the first and the third equations of Definition 5.1 we can deduce that

$$L(i)(\{f_T(x_1, \dots, x_n)\}_X) = \{((i + \text{id}) \bullet f_T)(L(i + \text{id})(x_1), \dots, L(i + \text{id})(x_n))\}_X$$

provided that  $i : T' \rightarrow U$  is an injective map,  $U \cap \text{bind}(f) = \emptyset$  and  $\text{id}$  is the identity map on  $\text{bind}(f)$ .

**Example 5.3.**

- 1 The functor  $\delta$  has a uniform presentation, with the operation symbols given in Section 4.1 structured as a presheaf as follows:

$$\begin{aligned} [a]_S &\in \mathcal{O}(S \cup \{a\}) \\ \mathcal{O}(w_b)([a]_S) &= [a]_{S \cup \{b\}} \\ \mathcal{O}((b/a)_S)([a]_S) &= [b]_S \end{aligned}$$

We have that  $\text{bind}([a]_S) = \{a\}$ , so the equations (20) are of the form (29).

- 2 The presentation of the functor used for the axiomatisation of  $\lambda$ -calculus, defined in (17), is also uniform. Indeed, the equations appearing in Proposition 4.7 are of the form (29), because  $\text{bind}(a_S) = \text{bind}(\text{app}_S) = \emptyset$ .
- 3 More generally, functors constructed from  $\mathcal{N}$ ,  $+$ ,  $\times$  and  $\delta$  have uniform presentations.

**Proposition 5.4.** If  $L$  has a uniform presentation, then:

- 1 there exists a natural transformation  $\alpha : LIR \rightarrow IRL$  such that the diagram below commutes:

$$\begin{array}{ccc} LIR & \xrightarrow{\alpha} & IRL \\ & \swarrow L\eta & \nearrow \eta L \\ & L & \end{array} \quad (30)$$

- 2  $\tilde{L} = RLI$  preserves sheaf epimorphisms.

*Proof.* Recall Remark 2.4, for the notation used for sheafification. We need to define a natural transformation  $\alpha$  making diagram (30) commutative. Let  $X$  be a presheaf. Elements of  $LIRX(T')$  will be of the form  $\{f_T(\overline{x}_1, \dots, \overline{x}_n)\}_{IRX}$ , where we can assume without loss of generality that  $x_1, \dots, x_n$  are elements supported by  $T$  of the same set  $X(S)$ , for some  $T \subseteq S$ .

We put  $\alpha_{T'}(\{f_T(\overline{x}_1, \dots, \overline{x}_n)\}_{IRX}) = \overline{\{f_S(x_1, \dots, x_n)\}_X}$ . In order to show that this is well-defined, we have to prove that  $\{f_S(x_1, \dots, x_n)\}_L$  is supported by  $T'$ . Let  $i, j : S \setminus \text{bind}(f) \rightarrow U$  be two injective maps that agree on  $T'$ . We have to show that

$$L(i)(\{f_S(x_1, \dots, x_n)\}_X) = L(j)(\{f_S(x_1, \dots, x_n)\}_X)$$

If  $U \cap \text{bind}(f) = \emptyset$  this follows easily from Remark 5.2 and the fact that the  $x_k$  are supported by  $T$ . If this is not the case, say for example, if  $U \cap \text{bind}(f) = \{a\}$ , then we can apply the second equation of Definition 5.1 for some name  $b \notin S \cup U$ :

$$\{f_S(x_1, \dots, x_n)\}_X = \{\mathcal{O}((b/a)_{S \setminus \{a\}})(f_S)((b/a)_{S \setminus \{a\}}x_1, \dots, (b/a)_{S \setminus \{a\}}x_n)\}_X$$

$$\boxed{
\begin{array}{c}
\frac{t_1 : T, \dots, t_n : T}{f(t_1, \dots, t_n) : T'} \quad (f : T \times \dots \times T \rightarrow T' \in \mathcal{O}(T)) \\
\\
\frac{t : T}{w_a t : T \uplus \{a\}} \quad \frac{t : T \uplus \{a\}}{(b/a)t : T \uplus \{b\}} \quad \frac{}{X : T_X}
\end{array}
}$$

Fig. 1. Uniform terms

and we can use again Remark 5.2, plus the fact that  $(b/a)_{S \setminus \{a\}} x_k$  is supported by  $T \setminus \{a\} \cup \{b\}$ .

It is now easy to see that  $\alpha$  makes diagram (16) commutative. It remains to check that  $\tilde{L}$  preserves sheaf epimorphisms. As noticed in the Remark 3.7, we know that  $LIR \simeq IRLIR$ , and this actually means that  $L$  preserves sheaves. Therefore, it is enough to prove that whenever  $e : X \rightarrow Y$  is a sheaf epimorphism,  $Lie : LIX \rightarrow LIY$  has the property stated in Proposition 2.3. Let  $y = \{f_S(y_1, \dots, y_n)\}_L$  be an element in  $(LIY)(S')$ , for some operation symbol  $f_S : S \times \dots \times S \rightarrow S'$  and  $y_1, \dots, y_n \in Y(S)$ . We prove that there exists an inclusion  $w' : S' \rightarrow T'$  and  $x \in LIX(T')$  such that  $LIY(w')(y) = (Lie)_{T'}(x)$ . Because  $e : X \rightarrow Y$  is a sheaf epimorphism, there exists an inclusion  $w : S \rightarrow T$  and  $x_k \in X(T)$  for all  $1 \leq k \leq n$  such that  $Y(w)(y_k) = e_T(x_k)$  for all  $k$ . Let  $w'$  denote the inclusion of  $S'$  into  $T' = T \setminus (S \setminus S')$  and let  $x \in LIX(T')$  be  $\{f_T(x_1, \dots, x_n)\}_{LX}$ . Using the first equation of Definition 5.1 we can derive  $LIY(w')(y) = (Lie)_{T'}(x)$ .  $\square$

**Corollary 5.5.** If  $L$  has a uniform presentation, then the HSP theorem 3.10 holds for  $\tilde{L}$ -algebras.

Looking at the theory of the  $\lambda$ -calculus in Example 4.8, we find that all operations are equivariant and that the equations are ‘uniform’ in  $S$ . To formalise the uniformity of an equational specification we first describe uniform terms, given by the set of rules in Figure 1.

In Figure 1, there are four schemas of rules: one for each operation  $f : T \times \dots \times T \rightarrow T'$ , two for the operations in  $\mathbb{I}$  (weakenings, substitutions), and one for variables. Each rule can be instantiated in an infinite number of ways:  $T$  ranges over finite sets of names and  $a, b$  over names. The notation  $T \uplus \{a\}$  indicates that an instantiation of the schema is only allowed for those sets  $T$  and those atoms  $a$  where  $a \notin T$ .

**Remark 5.6.** The rule for operations  $f(t_1, \dots, t_n)$  requires all arguments to be of the same type. This can be achieved by applying weakenings.

**Definition 5.7.** A signature given by a functor with a uniform presentation will be called **uniform signature**. A **uniform term**  $t : T$  for a uniform signature is a term  $t$  of type  $T$  formed according to the rules in Figure 1. A **uniform equation** is a pair of uniform terms of the same sort  $u = v : T$ , such that any variable  $X$  appears with the same type  $T_X$  in both  $u$  and  $v$ . A **uniform theory** consists of a set of uniform equations.

**Example 5.8.** A uniform theory for  $\lambda$ -calculus consists of the following uniform equations in the uniform signature given by the functor  $\delta + \mathcal{N} \times - + - \times -$ :

$$\mathbf{app}([a](w_a Y), X) = Y : \emptyset \quad (\beta-1)$$

$$\mathbf{app}([a]a, X) = X : \emptyset \quad (\beta-2)$$

$$\mathbf{app}([a]([b]_{\{a\}}(X)), Y) = [b](\mathbf{app}_{\{b\}}([a]_{\{b\}}(X), w_a Y)) : \emptyset \quad (\beta-3)$$

$$\mathbf{app}([a](\mathbf{app}_{\{a\}}(X, Y)), Z) = \mathbf{app}(\mathbf{app}([a](X), Z), \mathbf{app}([a](Y), Z)) : \emptyset \quad (\beta-4)$$

$$\mathbf{app}_{\{b\}}(w_b[a](X), b) = (b/a)X : \{b\} \quad (\beta-5)$$

$$[a](\mathbf{app}_{\{a\}}(w_a X, a)) = X : \emptyset \quad (\eta)$$

where  $\mathbf{app}$ ,  $[a]$  and  $a$  stand for  $\mathbf{app}_\emptyset$ ,  $[a]_\emptyset$  and  $a_\emptyset$ , respectively.

The idea is that a uniform equation  $u = v : T$  translates to a set of equations in the sense of standard many-sorted universal algebra:  $u_S = v_S : T \cup S$  where  $S$  ranges over the finite subsets of  $\mathcal{N}$  with  $S \cap T = \emptyset$ . If we want to extend the sort of the equation, we might also have to change the sort of the variables. There is a subtlety here: do we raise the type of the variables or do we add weakenings? We prefer the former if, for example, we want to raise the type of the equation  $X = Y : \emptyset$  by a set  $S$ . This becomes  $X_S = Y_S : S$ , where  $X_S, Y_S$  have type  $S$ .

Similarly, if we want to translate the equation (in which  $X : \emptyset$  has type  $\emptyset$ )

$$[a]w_a X = X : \emptyset \quad (31)$$

by a set  $\{b\}$ , where  $b$  is a different name than  $a$ , we should get

$$[a]w_a X_{\{b\}} = X_{\{b\}}$$

with  $X_{\{b\}}$  a variable of sort  $\{b\}$ . However, we should be able to translate (31) to a standard equation of sort  $\{a\}$ . We expect all the appearances of  $X$  within the translated equation to have the same sort. If, as above, we change the sort of  $X$  from  $\emptyset$  to  $\{a\}$ , then on the left hand side we would get  $[a]w_a X_{\{a\}}$ , and this does not match the arity of  $w_a$ . In this example, the left hand side of the equation has an implicit freshness constraint on the variable  $X$ . Because of the weakening  $w_a$  appearing in front of  $X$ , we will not be able to instantiate  $X$  with elements whose sorts contain  $a$ . So  $a$  is ‘fresh’ for  $X$ . The solution is to define the translation of this equation as

$$w_a[a]w_a X_\emptyset = w_a X_\emptyset : \{a\}$$

So we have to distinguish between the cases when we simply need to add some weakenings and the cases when we have to extend the sort of the variable. We formalise these observations in the next definitions.

**Definition 5.9.** The **freshness set** of a variable  $X$  appearing with sort  $T_X$  in an equation  $E$  of the form  $u = v : T$  is the set

$$\mathbf{Fr}_E(X) = \bigcup_{t:T} T \setminus T_X$$

where the union is taken over all sub-terms  $t$  of either  $u$  or  $v$  that contain the variable  $X$ .

**Example 5.10.** In the next uniform equation for the uniform signature given by the functor defined in (17),  $X$  has type  $\emptyset$  and  $\text{Fr}_E(X) = \{a\}$ .

$$[a]_{\emptyset} \mathbf{app}_{\{a\}}(w_a X, a_{\emptyset}) = X : \emptyset \quad (32)$$

**Definition 5.11.** The translation of an equation  $E$  of the form  $u = v : T_E$  by a finite set  $S$ , disjoint from  $T_E$ , is  $u_S = v_S : T \cup S$ , where the  $u_S = \text{tr}_S(u)$ ,  $v_S = \text{tr}_S(v)$  and translation  $\text{tr}_S(t : T)$  of a sub-term  $t$  of either  $u$  or  $v$  is defined by

$$\begin{aligned} \text{tr}_S(f_T(t_1, \dots, t_n) : T') &= w_{a_1} \dots w_{a_k} f_{T \cup S}(\text{tr}_{S \setminus T}(t_1), \dots, \text{tr}_{S \setminus T}(t_n)) \\ \text{tr}_S(w_a t : T \uplus \{a\}) &= w_{S \cup T, a} \text{tr}_S(t : T) \\ \text{tr}_S((b/a)t : T \uplus \{b\}) &= (b/a)_{S \cup T} \text{tr}_S(t : T \uplus \{a\}) && \text{if } a \notin S \\ \text{tr}_S((b/a)t : T \uplus \{b\}) &= w_a (b/a)_{S \setminus \{a\} \cup T} \text{tr}_{S \setminus \{a\}}(t : T \uplus \{a\}) && \text{if } a \in S \\ \text{tr}_S(X : T_X) &= w_{a_1} \dots w_{a_k} X_{T_X \cup S \setminus \text{Fr}_E(X)} \end{aligned} \quad (33)$$

where in the first condition  $f_T$  has arity  $T \times \dots \times T \rightarrow T'$  and  $\{a_1, \dots, a_k\} = S \cap \text{bind}(f)$ , see (28). In the last condition  $\{a_1, \dots, a_k\} = \text{Fr}_E(X) \cap S$  and  $X_{T_X \cup S \setminus \text{Fr}_E(X)}$  is a variable of sort  $T_X \cup S \setminus \text{Fr}_E(X)$ .

**Remark 5.12.** We only define  $\text{tr}_S(t : T)$  for finite sets  $S$  such that  $S \cap T = \emptyset$ . The above definition is sound, because initially we chose a set  $S$  disjoint from  $T_E$ , and then we can prove inductively that whenever we compute  $\text{tr}_{S'}(t : T')$  we have  $S' \cap T' = \emptyset$ .

Notice that the translation  $\text{tr}_S(t)$  depends on the equation for which  $t$  is a sub-term, and if  $t$  has sort  $T$  then the sort of  $\text{tr}_S(t)$  is  $S \cup T$ . In the first condition of (33), the set  $S$  is disjoint from  $T'$ , but may contain elements of  $\text{bind}(f)$ .

**Example 5.13.** For  $b \neq a$ , the translation by a set  $\{b\}$  of the uniform equation (32) is

$$[a]_{\emptyset} \mathbf{app}_{\{a\}}(w_a X_{\{b\}}, a_{\emptyset}) = X_{\{b\}} : \{b\}.$$

But the translation of the same equation by  $\{a, b\}$  is

$$w_a [a]_{\emptyset} \mathbf{app}_{\{a, b\}}(w_a X_{\{b\}}, a_{\{b\}}) = w_a X_{\{b\}} : \{a, b\}.$$

We can do this translation, because the set  $\{a, b\}$  is disjoint from the type of the uniform equation, which is the empty set. On the left hand side we use the weakening  $w_a$  because  $\{a, b\} \cap \text{bind}([a]_{\emptyset}) = \{a\}$ . On the right hand side we use the weakening  $w_a$  because  $\text{Fr}_E(X) \cap \{a, b\} = \{a\}$ . Also note that there is no connection between the variables  $X$ ,  $X_{\{a\}}$  and  $X_{\{a, b\}}$  appearing in the equations (32) and its translations by  $\{a\}$  and  $\{a, b\}$ , respectively. In particular, we should emphasize that we do not have that  $w_a X = X_{\{a\}}$ .

**Example 5.14.** Translating the uniform theory given in Example 5.8, we get a standard many-sorted theory equivalent to that given in Example 4.8. Apparently we obtain more equations, because, for instance, if we translate the  $(\beta-1)$  equation by the set  $\{a\}$  we get the equation

$$\mathbf{app}_{\{a\}}(w_a [a] w_a Y, X_{\{a\}}) = w_a Y$$

This does not appear among the equations of Example 4.8, but it can be derived from them.

The above definition of  $tr_S(X : T_X)$  is also justified by the next property that one expects for the set of standard equations obtained from a uniform equation.

**Lemma 5.15.** Let  $E$  be a uniform equation  $u = v : T$  in which the variable  $X$  appears with sort  $T_X$  and let  $S$  be a finite set of names disjoint from  $T$ . All the occurrences of  $X$  in the standard equation  $u_S = v_S : T \cup S$  have the same sort  $T_X \cup S \setminus \text{Fr}_E(X)$ .

*Proof.* Note that, while applying the algorithm described in Definition 5.11, the subscript of  $tr$  may change as we traverse the syntax tree of the terms. Explicitly, it will diminish by elements of  $\text{bind}(f)$  when we reach a term  $f_T(t_1, \dots, t_n)$  or by  $\{a\}$  if we reach  $(b/a)t$ . So, we may have to evaluate  $tr_{S'}(X : T_X)$  for some  $S' \subseteq S$ . We have to prove that  $T_X \cup S' \setminus \text{Fr}_E(X) = T_X \cup S \setminus \text{Fr}_E(X)$ . It suffices to show that  $S \setminus S' \subseteq T_X \cup \text{Fr}_E(X)$ . Let  $a$  be a name in  $S \setminus S'$ . Then there exists a term  $t$ , containing  $X$  such that either  $(b/a)t$  or  $f_T(t_1, \dots, t, \dots, t_n)$ , for some operation symbol  $f_T$  with  $a \in \text{bind}(f)$ , is a subterm of either  $u$  or  $v$ . In both cases  $t$  is a sub-term within the equation  $E$  that contains  $X$  and such that  $S' \cup \{a\}$  is included in the sort of  $t$ . Using the Definition 5.9, we either have that  $a \in T_X$  or  $a \in \text{Fr}_E(X)$ .  $\square$

**Notation 5.16.** A sheaf algebra  $\mathbb{A}$  for a uniform signature given by a functor  $L$  is an  $L$ -algebra  $\alpha : LA \rightarrow A$ . In subsequent proofs and calculations, we will use the following notation: for  $f_T \in \Sigma$  having arity  $T \times \dots \times T \rightarrow T'$ ,  $\alpha$  maps  $(x_1, \dots, x_n) \in A^n(T)$  to  $f_T^\mathbb{A}(x_1, \dots, x_n) \in A(T')$ . For each algebra  $\mathbb{A}$  and each valuation  $v$  sending variables  $X : T_X$  to elements of  $A(T_X)$ , a term  $t : T$  of type  $T$  evaluates to an element  $\llbracket t \rrbracket_{\mathbb{A}, v, T}$  in  $A(T)$ .

**Definition 5.17.** An algebra  $\mathbb{A}$  satisfies the uniform equation  $t = u : T$  iff for all  $S \cap T = \emptyset$  and all valuations  $v$  of variables, we have that  $\mathbb{A}, v \models t_S = u_S$ , that is,  $\llbracket t \rrbracket_{\mathbb{A}, v, S \cup T}$  and  $\llbracket u \rrbracket_{\mathbb{A}, v, S \cup T}$  denote the same element of  $A(S \cup T)$ .

In the remainder of the section, we are going to show that classes of sheaf algebras defined by uniform equations are precisely those closed under sheaf quotients, sub-algebras, products and abstraction. In our setting, abstraction (which corresponds to atoms-abstraction (Gabbay, 2008)) maps an algebra with carrier  $A$  to an algebra with carrier  $\delta A$ . To describe this notion, we need to recall the definition of  $\delta$  from Section 4.1. For  $c \notin S$ , there is an isomorphism

$$\begin{aligned} A(S \cup \{c\}) &\rightarrow \delta A(S) \\ x &\mapsto \{[c]_S x\}_{\delta A} \end{aligned} \tag{34}$$

**Definition 5.18.** Given a nominal algebra  $\mathbb{A}$  for a uniform signature with structure  $LA \rightarrow A$  its abstraction  $\delta \mathbb{A}$  with structure  $L(\delta A) \rightarrow \delta A$  is given by

$$f_T^{\delta \mathbb{A}}(\{[c]x_1\}_{\delta \mathbb{A}}, \dots, \{[c]x_n\}_{\delta \mathbb{A}}) = \{[c]f_{T \cup \{c\}}^\mathbb{A}(x_1, \dots, x_n)\}_{\delta \mathbb{A}}$$

where  $c \notin T$ .

The next lemmas establish a connection between the evaluation of a uniform term  $t : T$  in  $\delta \mathbb{A}$  and the evaluation of  $t_{\{a\}} : T \cup \{a\}$  in  $\mathbb{A}$ , for  $a \notin T$ . Note that this is possible for

uniform terms, but not for terms. Recall from Definition 5.7 that a uniform term is not a term (in the sense of set-based universal algebra) but a family of terms.

**Lemma 5.19.** Consider a uniform subterm  $t : T$  within an equation  $E$ . For all atoms  $a \notin T$  and for all valuations  $v_{\mathbb{A}}$  in  $\mathbb{A}$  of the variables in  $t_{\{a\}} = tr_{\{a\}}(t)$ , there exists a valuation  $v_{\delta\mathbb{A}}$  in  $\delta\mathbb{A}$  of the variables in  $t$  such that

$$\llbracket t \rrbracket_{\delta\mathbb{A}, v_{\delta\mathbb{A}}, T} = \{[a]_T \llbracket t_{\{a\}} \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T \cup \{a\}}\}_{\delta A} \quad (35)$$

*Proof.* Note that a variable  $X : T_X$  in  $t$  will have sort  $T_X \cup \{a\} \setminus \text{Fr}_E(X)$  in  $t_{\{a\}}$ . We know that  $v_{\mathbb{A}}(X) \in A(T_X \cup \{a\} \setminus \text{Fr}_E(X))$ . We define  $v_{\delta\mathbb{A}}(X) \in \delta A(T_X)$  by

$$v_{\delta\mathbb{A}}(X) = \begin{cases} \{[a]_T v_{\mathbb{A}}(X)\}_{\delta A} & \text{if } a \notin T_X \cup \text{Fr}_E(X) \\ \{[b]_A v_{\mathbb{A}}(X)\}_{\delta A} & \text{if } a \in T_X \cup \text{Fr}_E(X) \end{cases}$$

We can prove that if  $a$  belongs to the sort  $U$  of a subterm  $u : U$  of  $t$ , then

$$\llbracket u \rrbracket_{\delta\mathbb{A}, v_{\delta\mathbb{A}}, U} = \{[b]_U w_b \llbracket u \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, U}\}_{\delta A} \quad (36)$$

Now we can prove (35) by induction on the structure of terms. For example, let  $f_T : T \times \dots \times T \rightarrow T'$  be an operation symbol such that  $a \notin T'$ . If  $a \notin T$  then the proof follows by induction. But, if  $a \in T \setminus T'$ , then we have

$$\begin{aligned} \llbracket f_T(t_1, \dots, t_n) \rrbracket_{\delta\mathbb{A}, v_{\delta\mathbb{A}}, T'} &= f_T^{\delta\mathbb{A}}(\llbracket t_1 \rrbracket_{\delta\mathbb{A}, v_{\delta\mathbb{A}}, T}, \dots, \llbracket t_n \rrbracket_{\delta\mathbb{A}, v_{\delta\mathbb{A}}, T}) \\ &= f_T^{\delta\mathbb{A}}(\{[b]_T w_b \llbracket t_1 \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T}\}_{\delta A}, \dots, \{[b]_T w_b \llbracket t_n \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T}\}_{\delta A}) \\ &= \{[b]_T w_b f_T^{\mathbb{A}}(\llbracket t_1 \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T}, \dots, \llbracket t_n \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T})\}_{\delta A} \\ &= \{[a]_T w_a f_T^{\mathbb{A}}(\llbracket t_1 \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T}, \dots, \llbracket t_n \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T})\}_{\delta A} \\ &= \{[a]_T f_T^{\mathbb{A}}(\llbracket t_1 \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T}, \dots, \llbracket t_n \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T})\}_{\delta A} \\ &= \{[a]_T \llbracket t_{\{a\}} \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, T \cup \{a\}}\}_{\delta A} \end{aligned}$$

□

For illustration, consider the uniform term  $t = \text{app}_{\emptyset}(X, Y)$  within an equation  $E$ , such that  $X, Y$  are variables of sort  $\emptyset$ , with  $\text{Fr}_E(X) = \{a\}$  and  $\text{Fr}_E(Y) = \emptyset$ . We have that  $t_{\{a\}} = \text{app}_{\{a\}}(w_a X, Y_{\{a\}})$ . Then  $v_{\delta\mathbb{A}}(X) \in \delta\mathbb{A}(\emptyset)$  is defined as  $v_{\delta\mathbb{A}}(X) = \{[b]_A v_{\mathbb{A}}(X)\}_{\delta A}$ , for some fresh  $b$ , whilst  $v_{\delta\mathbb{A}}(Y) \in \delta\mathbb{A}(\emptyset)$  is defined as  $v_{\delta\mathbb{A}}(Y) = \{[a]_A v_{\mathbb{A}}(Y_{\{a\}})\}_{\delta A}$ .

**Lemma 5.20.** Consider a uniform term  $t : T$  within an equation  $E$  and let  $a$  be a name such that  $a \notin T_X \cup \text{Fr}_E(X)$  for all variables  $X$  occurring in  $E$ . For all valuations  $v_{\delta\mathbb{A}}$  in  $\delta\mathbb{A}$  of the variables in  $t$  there exists a valuation in  $\mathbb{A}$  of the variables in  $tr_{\{a\}}(t)$ , such that (35) holds.

*Proof.* Note that if  $X$  has type  $T_X$  in  $t$ , it has type  $T_X \cup \{a\} \setminus \text{Fr}_E(X) = T_X \cup \{a\}$  in  $tr_{\{a\}}(t)$ . We define  $v_{\mathbb{A}}(X)$  as the unique element of  $A(T_X \cup \{a\})$  such that  $v_{\delta\mathbb{A}}(X) = \{[a]_{T_X} v_{\mathbb{A}}(X)\}_{\delta A}$ . The proof is by induction on the structure of terms. □

**Proposition 5.21.** If a class  $\mathcal{B}$  of nominal algebras is defined by a uniform set of equations, then  $\mathcal{B}$  is closed under abstraction.

*Proof.* Assume that the nominal algebra  $\mathbb{A}$  satisfies a uniform equation  $t = u : T$ .

Consider a valuation  $v$  of the variables of  $t_S, u_S$  in the algebra  $\delta\mathbb{A}$ . We need to show  $\delta\mathbb{A}, v \models t_S = u_S$  for all finite sets of names  $S$ , disjoint from  $T$ . Choose a name  $a$  such that  $a \notin S \cup T$  and  $a \notin T_X \cup \text{Fr}_E(X)$  for all variables  $X$ . Consider the valuation  $v_{\mathbb{A}}$  as in Lemma 5.20. Since  $\mathbb{A}$  satisfies  $t_{S \cup \{a\}} = u_{S \cup \{a\}} : T \cup S \cup \{a\}$ , we have  $\llbracket t_{S \cup \{a\}} \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, S \cup T \cup \{a\}} = \llbracket u_{S \cup \{a\}} \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, S \cup T \cup \{a\}}$ . Then  $\llbracket t_S \rrbracket_{\delta\mathbb{A}, v, S \cup T} = \llbracket u_S \rrbracket_{\delta\mathbb{A}, v, S \cup T}$  follows from Lemma 5.20 applied for  $t_S$  and  $u_S$ .  $\square$

**Proposition 5.22.** If a class  $\mathcal{B}$  of nominal algebras is defined by a uniform set of equations, then  $\mathcal{B}$  is closed under quotients.

*Proof.* Consider a quotient of sheaves  $f : \mathbb{A} \rightarrow \mathbb{B}$ , such that  $\mathbb{A}$  satisfies the uniform equations. Consider the uniform equation  $t = u : T$  and choose  $S$  disjoint from  $T$  and a valuation  $v$  in  $\mathbb{B}$  of the variables in  $t_S$  and  $u_S$ . We have to show  $\llbracket t_S \rrbracket_{\mathbb{B}, v, S \cup T} = \llbracket u_S \rrbracket_{\mathbb{B}, v, S \cup T}$ . If a variable  $X$  has sort  $T_X$  in the uniform equation, its translation has sort  $T_X \cup S \setminus \text{Fr}_E(X)$  in  $t_S = u_S$ , so  $v(X) \in \mathbb{B}(T_X \cup S \setminus \text{Fr}_E(X))$ . Using Proposition 2.3, we can find a finite set of names  $S'$ , such that  $S \subseteq S'$  and for all variables  $X$  appearing in the equation there exists  $v_{\mathbb{A}}(X) \in A(T_X \cup S' \setminus \text{Fr}_E(X))$  such that  $f_{T_X \cup S' \setminus \text{Fr}_E(X)}(v_{\mathbb{A}}(X)) = \mathbb{B}(w_X)(v(X))$ , where  $w_X$  denotes the inclusion  $w_X : T_X \cup S \setminus \text{Fr}_E(X) \rightarrow T_X \cup S' \setminus \text{Fr}_E(X)$ .  $v_{\mathbb{A}}$  is a valuation of the variables in  $t_{S'} = u_{S'}$ . From this, we prove by induction on the structure of  $t$  that  $f_{S' \cup T}(\llbracket t_{S'} \rrbracket_{\mathbb{A}, v_{\mathbb{A}}, S' \cup T}) = B(w)(\llbracket t_S \rrbracket_{\mathbb{B}, v, S \cup T})$ . Since  $B(w)$  is injective, this concludes the proof.  $\square$

**Theorem 5.23.** A class  $\mathcal{B}$  of sheaf algebras for a uniform signature is definable by uniform equations if and only if it is closed under sheaf quotients, sub-algebras, products and abstraction.

*Proof.* Assume that a class of nominal algebras for a uniform signature is defined by uniform equations. Using Corollary 5.5, we can derive closure under subalgebras and products. Closure under abstraction and sheaf quotients follows from Propositions 5.21 and 5.22, respectively. Conversely, from closure under HSPA we derive closure under presheaf epimorphisms, subalgebras and products, hence the class of sheaf-algebras is definable by a set of equations  $\mathcal{E}$  in the sense of standard many-sorted universal algebra. We have to show that these equations come from a uniform theory. It is enough to show that whenever  $t = u : T$  is in  $\mathcal{E}$ , then for all  $a \notin T$  and  $B \in \mathcal{B}$  we have that  $B \models t_{\{a\}} = u_{\{a\}} : T \cup \{a\}$ . This follows from closure under abstraction and Lemma 5.19.  $\square$

## 6. Comparison with other nominal logics

**Preliminaries on nominal sets** We briefly recall the definition of a nominal set. Intuitively, this is a set equipped with an additional structure which allows well-behaved name swapping in elements of the set. A left action of the group  $\mathcal{S}(\mathcal{N})$  of all finitely supported permutations of the set of names  $\mathcal{N}$  is a pair  $(|\mathbb{X}|, \cdot)$  consisting of a set  $|\mathbb{X}|$  and

a function  $\cdot : \mathbf{S}(\mathcal{N}) \times |\mathbb{X}| \rightarrow |\mathbb{X}|$  satisfying:

$$\begin{aligned} \text{id}_{\mathcal{N}} \cdot x &= x \\ (\sigma\tau) \cdot x &= \sigma \cdot (\tau \cdot x) \end{aligned} \tag{37}$$

for all  $x \in |\mathbb{X}|$  and  $\sigma, \tau \in \mathbf{S}(\mathcal{N})$ . Let  $x$  be an element of  $|\mathbb{X}|$ . We say that a subset  $S \subseteq \mathcal{N}$  *supports*  $x$  if and only if for all  $a, b \in \mathcal{N} \setminus S$  we have that  $(a, b) \cdot x = x$ , where  $(a, b)$  denotes the transposition which swaps  $a$  and  $b$ . The element  $x$  is said to be *finitely supported* if and only if there exists a finite set  $S$  which supports  $x$ .

**Definition 6.1.** A nominal set is a left  $\mathbf{S}(\mathcal{N})$ -action  $(|\mathbb{X}|, \cdot)$  such that each element of  $x$  is finitely supported.

One can check that for each element  $x$  of a nominal set there exists a smallest set, in the sense of inclusion, which supports  $x$ . This set is called the *support* of  $x$  and will be denoted by  $\text{supp}(x)$ . We say that  $a \in \mathcal{N}$  is *fresh* for  $x$ , if  $a \notin \text{supp}(x)$ . A morphism of nominal sets  $f : (|\mathbb{X}|, \cdot) \rightarrow (|\mathbb{Y}|, \circ)$  is an *equivariant* function between the carrier sets, meaning that  $f$  behaves well with respect to permutations of names:  $f(\sigma \cdot x) = \sigma \circ f(x)$  for all  $x \in |\mathbb{X}|$ .

We will denote by  $\mathbf{Nom}$  the category of nominal sets and equivariant maps.

**Example 6.2.** The set  $\mathcal{N}$  equipped with the action given by evaluation,  $\sigma \cdot a = \sigma(a)$ , is a nominal set.

**Remark 6.3.** If  $\mathcal{N}$  and  $|\mathbb{X}|$  are equipped with the discrete topology, then  $\mathbf{S}(\mathcal{N})$  can be equipped with the topology induced by the product topology on  $\mathcal{N}^{\mathcal{N}}$ . Then a nominal set  $(|\mathbb{X}|, \cdot)$  is just a continuous  $\mathbf{S}(\mathcal{N})$ -action, i.e.  $\cdot$  is a continuous function.

The equivalence between the nominal sets and the Schanuel topos is a corollary of (MacLane and Moerdijk, 1994, Theorem III.9.1). Spelling out the proof of this theorem we get

**Remark 6.4.** If  $(|\mathbb{X}|, \cdot)$  is a nominal set the corresponding sheaf  $\mathfrak{X} \in \mathbf{Sh}(\mathbb{I}^{op})$  is obtained by taking

$$\begin{aligned} \mathfrak{X}(S) &= \{x \in |\mathbb{X}| \mid \text{supp}(x) \subseteq S\} \\ \mathfrak{X}(w_a)(x) &= x & x \in \mathfrak{X}, w_a : S \rightarrow S \cup \{a\} \\ \mathfrak{X}(b/a)_S(x) &= (a, b) \cdot x & x \in S \cup \{a\} \end{aligned} \tag{38}$$

Note that the inclusion functor  $\mathbb{I} \hookrightarrow \mathbf{Set}$  corresponds precisely to the nominal set  $(\mathcal{N}, \cdot)$  described in Example 6.2.

### 6.1. Comparison with nominal algebra

In this subsection we show how to translate the syntax and theories of the nominal algebra (Gabbay and Mathijssen, 2009) to uniform signatures and uniform theories. Then we prove semantic invariance, that is, there is a correspondence between models for a nominal algebra theory and models for the uniform theory obtained via this translation.

**Translation of syntax.** We refer the reader to (Gabbay and Mathijssen, 2009) for the syntax and semantics of nominal algebra. To each nominal algebra signature corresponds a uniform signature given by the functor  $\mathcal{N} + \delta + \Sigma$ , where  $\delta$  is as in Section 4 and  $\Sigma$  is a polynomial functor on  $\mathbf{Set}^{\mathbb{I}}$ , given by

$$\Sigma A = \coprod \{f_{n_i}\} \times A^{n_i}$$

where the coproduct is taken after all the operation symbols  $f_{n_i}$  with arity  $n_i$  in the nominal algebra signature.

**Translations of equational judgements.** Assume  $\Delta \vdash t = u$  is an equality judgement in the sense of (Gabbay and Mathijssen, 2009). It is reasonable to ask that the uniform equation obtained by translating such an equality judgement has to satisfy the following:

- 1 All occurrences of  $X$  in the uniform equation have the same sort.
- 2 If  $a\#X$  is in  $\Delta$ , then in the uniform equation we can only instantiate  $X$  with elements whose support does not contain  $a$ .
- 3 We can prove semantic invariance of this translation, that is, a nominal set satisfies an equational judgement iff the corresponding sheaf satisfies the translated uniform equation.

In order to address 1. and 2. above, for each unknown  $X$  appearing in this judgement, we have to consider the following sets:  $\mathbf{anc}(X)$  defined as the set of names  $a$  for which there is an occurrence of  $\pi X$ , for some permutation  $\pi$ , such that  $[a]$  is an ancestor of  $\pi X$  in the syntax tree of the equation,<sup>‡</sup> and

$$\mathbf{fresh}(X) = \{a \in \mathcal{N} \mid a\#X \in \Delta\}.$$

Before giving the actual translation, we will first find the type  $T_E$  of the uniform equation  $E$ , obtained by translating  $\Delta \vdash t = u$ . This is done recursively:

$$\begin{aligned} \mathbf{type}(t = u) &= \mathbf{type}(t) \cup \mathbf{type}(u) \\ \mathbf{type}(f(t_1, \dots, t_n)) &= \cup \mathbf{type}(t_i) \\ \mathbf{type}([a]t) &= \mathbf{type}(t) \setminus \{a\} \\ \mathbf{type}(a) &= \{a\} \\ \mathbf{type}(\pi X) &= (\mathbf{anc}(X) \setminus \mathbf{fresh}(X)) \cup \mathbf{supp}(\pi) \end{aligned} \tag{39}$$

We define

$$T_E = \mathbf{type}(t = u) \cup \left( \bigcup_{X \in E} (\mathbf{fresh}(X) \setminus \mathbf{anc}(X)) \right).$$

The reason for adding  $\bigcup_{X \in E} (\mathbf{fresh}(X) \setminus \mathbf{anc}(X))$ , is that we want to be able to retrieve the names in  $\mathbf{fresh}(X)$  from the uniform equation obtained, even if they do not appear in any subterms. For example, the type of the translation of  $b\#X \vdash X = Y$  should be  $\{b\}$ , and not the empty set.

<sup>‡</sup> We say that  $[a]$  is an ancestor of  $\pi X$  rather than of  $X$ , because, in the definition of nominal terms,  $X$  is not a nominal subterm of the moderated unknown  $\pi X$ .

The actual translation is the uniform equation  $\mathcal{T}_{T_E}(t) = \mathcal{T}_{T_E}(u) : T_E$ , where  $\mathcal{T}_T(t) : T$  is a uniform term of type  $T$ , defined recursively by:

$$\begin{aligned} \mathcal{T}_T(f(t_1, \dots, t_n)) &= f(\mathcal{T}_T(t_1), \dots, \mathcal{T}_T(t_n)) : T \\ \mathcal{T}_T(a) &= a_T : T \\ \mathcal{T}_T([a]t) &= [a]_T \mathcal{T}_{T \cup \{a\}}(t) : T && \text{if } a \notin T \\ \mathcal{T}_T([a]t) &= w_a [a]_T \mathcal{T}_T(t) : T && \text{if } a \in T \\ \mathcal{T}_T(\pi X) &= \pi_T w_{a_1} \dots w_{a_k} X_{T \setminus \text{fresh}(X)} : T \end{aligned} \quad (40)$$

where, in the last condition  $\{a_1, \dots, a_k\} = T \cap \text{fresh}(X)$ . On the right hand side of the above equations, we have nominal terms for the uniform signature given by  $\mathcal{N} + \delta + \Sigma$ , obtained according to the rules in Figure 1.

**Example 6.5.** Consider the following judgement in nominal algebra:

$$a \# X \vdash [a] \text{app}(X, a) = X$$

We have that  $\text{fresh}(X) = \{a\}$ ,  $\text{anc}(X) = \{a\}$  and that the type of the translated uniform equation is  $\emptyset$ . The translation is the uniform equation

$$[a]_{\emptyset} \text{app}_{\{a\}}(w_a X, a_{\emptyset}) = X$$

and this corresponds to the set of equations  $(\eta)$  of Example 4.8, that is indexed after all finite sets  $S$  that do not contain  $a$ .

**Lemma 6.6.** If  $X$  is an unknown appearing in the equality judgement  $E$ , then all the instances of the variable  $X$  have the same sort  $(T_E \cup \text{anc}(X)) \setminus \text{fresh}(X)$  in the translated uniform equation.

*Proof.*  $X$  may appear more than once in the equality judgement. When traversing the syntax tree, the subscript of  $\mathcal{T}$  may change, so we have to prove that whenever we have to translate  $\mathcal{T}_T(\pi X)$ , the set  $T$  has the property that  $T \setminus \text{fresh}(X) = (T_E \cup \text{anc}(X)) \setminus \text{fresh}(X)$ . Note that first we apply the translation with index  $T_E$ , and as we traverse the tree, this sort will only increase by a name  $a$  when we reach a subterm of the form  $[a]t$ . If we eventually reach a leaf containing the unknown  $X$ , such an  $a$  must be in the set  $\text{anc}(X)$ . Therefore  $T \subseteq T_E \cup \text{anc}(X)$ , so we know that  $T \setminus \text{fresh}(X) \subseteq (T_E \cup \text{anc}(X)) \setminus \text{fresh}(X)$ . Conversely, let  $a \in (T_E \cup \text{anc}(X)) \setminus \text{fresh}(X)$ . If  $a \in T_E \setminus \text{fresh}(X)$ , then  $a \in T \setminus \text{fresh}(X)$  because  $T_E \subseteq T$ . It remains to consider the case when  $a \in \text{anc}(X) \setminus \text{fresh}(X)$ . We distinguish two cases, depending on whether this particular instance of  $X$  has  $[a]$  as an ancestor. If this is the case, the set  $T$  must contain the name  $a$ . If this is not the case, then we have that  $a \in \text{type}(t = u) \subseteq T_E$ , hence  $a \in T$ .  $\square$

**Lemma 6.7.**  $\text{Fr}_E(X) = \text{fresh}(X)$ .

*Proof.* Consider  $a \in \text{fresh}(X)$ . Let us denote by  $T_X$  the sort of the variable  $X$  in  $\mathcal{T}(E)$ . We know that  $T_X = (T_E \cup \text{anc}(X)) \setminus \text{fresh}(X)$ , so  $a \notin T_X$ . We have two cases:

- 1 If  $a \in \text{anc}(X)$ , then there exists a subterm  $[a]v$ , such that  $X$  occurs in  $v$ . The sort of  $\mathcal{T}(v)$  must contain the name  $a$ , so  $a \in \text{Fr}_E(X)$ .
- 2 If  $a \in \text{fresh}(X) \setminus \text{anc}(X)$ , then  $a \in T_E$ , so again we obtain that  $a \in \text{Fr}_E(X)$ .

Conversely, if  $a \in \text{Fr}_E(X)$  there exists a subterm  $v$  in  $E$  containing  $\pi X$ , for some permutation  $\pi$ , such that the variable  $X : T_X$  occurs in  $\mathcal{T}_T(v) : T$  and  $a \in T \setminus T_X$ . If the sort of  $\mathcal{T}(\pi X)$  is  $S$ , we have that  $T \subseteq S$ , hence we get that  $a \in S \setminus T_X = S \cap \text{fresh}(X)$ , so  $a \in \text{fresh}(X)$ .  $\square$

**Translation of semantics.** Let  $\mathbb{X} = (|\mathbb{X}|, \cdot, \mathbb{X}_{\text{atm}}, \mathbb{X}_{\text{abs}}, \{\mathbb{X}_f \mid f \in \mathbb{S}\})$  be a nominal algebra for a nominal signature  $\mathbb{S}$ . Let  $\mathcal{N} + \delta + \Sigma$  be the functor corresponding to this signature. We consider the sheaf  $\mathfrak{X}$  obtained from the nominal set  $(|\mathbb{X}|, \cdot)$  as in (38). The translation of  $\mathbb{X}$  is the sheaf algebra  $\mathcal{N} + \delta\mathfrak{X} + \Sigma\mathfrak{X} \rightarrow \mathfrak{X}$  given by

$$\begin{aligned} a_S &\mapsto \mathbb{X}_{\text{atm}}(a) \\ \{[a]_S x\}_{\delta\mathfrak{X}} &\mapsto \mathbb{X}_{\text{abs}}(\mathbb{X}_{\text{atm}}(a), x) \\ (f, x_1, \dots, x_n) &\mapsto \mathbb{X}_f(x_1, \dots, x_n) \end{aligned}$$

That this is well defined follows from the equivariance of  $\mathbb{X}_{\text{atm}}, \mathbb{X}_{\text{abs}}, \mathbb{X}_f$ .

**Theorem 6.8.** [semantic invariance] Let  $\mathbb{X}$  be a nominal algebra for a nominal signature. Let  $E$  be the uniform equation of type  $T_E$ , obtained by translating an equality judgement  $\Delta \vdash u = v$ . Then  $\mathbb{X}$  satisfies  $\Delta \vdash u = v$  if and only if  $\mathfrak{X}$  satisfies the uniform equation  $E$ .

*Proof.* First assume that  $\llbracket \Delta \vdash u = v \rrbracket^{\mathbb{X}}$  holds. We need to prove that  $\mathfrak{X} \models \text{tr}_S E$  for all finite sets  $S$  that are disjoint from  $T_E$ . Consider a valuation  $\varsigma$  of the variables appearing in  $\text{tr}_S(E)$  in  $\mathfrak{X}$ . If  $X$  is a variable of sort  $T_X$  in  $E$ , then  $X$  has sort  $S \setminus \text{Fr}_E(X) \cup T_X$  in  $\text{tr}_S(E)$ . So  $\varsigma(X) \in \mathfrak{X}(S \setminus \text{Fr}_E(X) \cup T_X)$  is an element of the nominal set  $\mathbb{X}$  supported by  $S \setminus \text{Fr}_E(X) \cup T_X$ . Using Lemma 6.6 and Lemma 6.7, we know that  $S \setminus \text{Fr}_E(X) \cup T_X$  is disjoint from  $\text{fresh}(X)$ , so whenever the freshness primitive  $a\#X$  is in  $\Delta$ ,  $a\#\varsigma(X)$ . Let  $\varsigma'$  be a valuation in  $\mathbb{X}$  of the unknowns in  $\Delta \vdash u = v$  that maps  $X$  to  $\varsigma(X)$ . We have to prove that  $\llbracket \text{tr}_S(\mathcal{T}u) \rrbracket_{\varsigma}^{\mathfrak{X}} = \llbracket \text{tr}_S(\mathcal{T}v) \rrbracket_{\varsigma}^{\mathfrak{X}}$ . This follows from the claim below, which can be proved by induction on the structure of the terms.

**Claim 6.9.** For all subterms  $t$  of either  $u$  or  $v$ , we have  $\llbracket t \rrbracket_{\varsigma'}^{\mathbb{X}} = \llbracket \mathcal{T}t \rrbracket_{\varsigma}^{\mathfrak{X}}$ .

Conversely, assume that  $\mathfrak{X}$  satisfies the uniform equation  $E$ . Consider a valuation  $\varsigma'$  in  $\mathbb{X}$  of the unknowns of  $\Delta \vdash u = v$ , such that  $a\#\varsigma'(X)$  whenever  $a\#X \in \Delta$ . We consider the finite set of atoms  $S := \bigcup_X \text{supp}(\varsigma'(X)) \setminus T_E$ . We can define a valuation  $\varsigma$  of the variables occurring in  $\text{tr}_S(E)$  in  $\mathfrak{X}$ , simply by taking  $\varsigma(X) = \varsigma'(X)$ . In order to prove that this is well defined we can check that  $\varsigma'(X)$  is supported by  $S \setminus \text{Fr}_E(X) \cup T_X$ . Since  $\varsigma'$  can be obtained from  $\varsigma$  as before, we can finalize the proof by applying again the Claim 6.9.  $\square$

**Corollary 6.10.** Theorem 6.8 and Theorem 5.23 give a new proof for Gabbay's HSPA theorem, (Gabbay, 2008, Theorem 9.3).

## 6.2. Comparison with NEL

This section compares uniform theories to a fragment of the nominal equational logic of (Clouston and Pitts, 2007). For simplicity we only consider the one-sorted version of

NEL,<sup>§</sup> although extending our work to many-sortedness over sheaves is not difficult. We only consider theories for which the axioms are of the form

$$\begin{aligned} \{X_1, \dots, X_n\} \vdash \bar{a} \# f(X_1, \dots, X_n) \quad \bar{a} \subseteq \text{supp}(f) \\ \Delta \vdash t \approx t' \end{aligned} \tag{41}$$

**Translation of syntax.** Recall that a signature for NEL is given by a nominal set  $\text{Op} = (|\text{Op}|, \cdot)$  of operation symbols. Consider a theory for this signature, consisting of axioms as in (41). We will construct a presheaf of operations  $\mathcal{O}$  as in Definition 5.1, which is almost the sheaf corresponding to  $\text{Op}$  via the isomorphism between nominal sets and  $\text{Sh}(\mathbb{I}^{op})$ : for all finite sets of names  $S$ ,  $\mathcal{O}(S)$  contains the operation symbols whose support is contained in  $S$ . But we also add more information about the arity of these operation symbols.

**Definition 6.11.** Consider a NEL theory for a signature that contains an operation symbol  $f$ . The set  $\text{bind}(f)$  is defined as the set of names  $a$  such that there is an axiom in the theory of the form  $\{X_1, \dots, X_n\} \vdash \bar{b} \# (\pi \cdot f)(X_1, \dots, X_n)$  for a finite set of names  $\bar{b}$  and a permutation  $\pi$ , such that  $\pi(b) = a$  for some  $b \in \bar{b}$ .

If  $f \in \text{Op}$  is an  $n$ -ary operation symbol, such that  $\text{supp}(f) = T$  and  $T \subseteq S$ , we consider an operation symbol in  $f_S \in \mathcal{O}(S)$ , with arity  $f_S : S \times \dots \times S \rightarrow S \setminus \text{bind}(f)$ . The definition above implies that  $\text{bind}(\pi \cdot f) = \pi[\text{bind}(f)]$ . We also obtain that  $\text{bind}(f) \subseteq \text{supp}(f)$ . So we can derive that, for any injective map  $j : S \rightarrow S'$ ,  $\mathcal{O}(j)(f_S)$  has the arity  $S' \times \dots \times S' \rightarrow S' \setminus j[\text{bind}(f)]$ . So  $\mathcal{O}$  is a presheaf as in Definition 5.1. Note that the arity of an operation symbol in  $\mathcal{O}$  depends not only on the nominal signature, but also on the theory, because of the way freshness constraints are expressed in NEL, see (41). The translation of a NEL signature is the uniform signature given by the functor  $L$  with a uniform presentation given by  $\mathcal{O}$ : see Definition 5.1.

**Example 6.12.** If  $L_a$  is an operation symbol as in the NEL signature for  $\lambda$ -calculus of (Clouston and Pitts, 2007, Example 3.1) then  $\text{bind}(L_a) = \{a\}$ . The translation of this NEL signature is the uniform signature given by the functor defined in (17).

**Translation of a theory.** From each axiom in a theory in the sense of (Clouston and Pitts, 2007), having the form

$$\Delta \vdash t \approx t'$$

we will obtain a uniform equation  $E$  of sort  $T_E$ . As in the previous section we first describe a way of finding the sort  $T_E$ . Again, all occurrences of a variable  $X$  are expected to have the same sort in the translation, so we need to pay attention to the bound names of the terms that contain  $X$  and to the names that should be fresh for  $X$ . To this end we define the set  $\text{anc}(X)$  by

$$\text{anc}(X) = \bigcup \text{bind}(f)$$

<sup>§</sup> Note that we will not need the sorting environments of (Clouston and Pitts, 2007) in this case.

taken after all operations  $f$  such that  $X$  appears in a subterm of either  $t$  or  $t'$  of the form  $f(t_1, \dots, t_n)$ . Similarly we define

$$\text{fresh}(X) = \bar{a} \quad \text{iff} \quad \bar{a} \not\# X \in \Delta$$

The fact that  $a \not\# X$  is in the freshness environment will be expressed in the uniform equation by adding a weakening  $w_a$  in front of  $X$ .

In order to find  $T_E$  we define a function  $\text{type}$  recursively:

$$\begin{aligned} \text{type}(t = u) &= \text{type}(t) \cup \text{type}(u) \\ \text{type}(f(t_1, \dots, t_n)) &= (\cup \text{type}(t_i) \cup \text{supp}(f)) \setminus \text{bind}(f) \\ \text{type}(\pi X) &= (\text{anc}(X) \setminus \text{fresh}(X)) \cup \text{supp}(\pi) \end{aligned} \quad (42)$$

We define  $T_E = \text{type}(t = u) \cup (\bigcup_{X \in E} (\text{fresh}(X) \setminus \text{anc}(X)))$

The translation of the axiom  $\Delta \vdash t \approx t'$  is the uniform equation  $\mathcal{T}_{T_E}(t) = \mathcal{T}_{T_E}(u) : T_E$ , where  $\mathcal{T}_T(t)$  is a term of sort  $T$ , defined recursively by

$$\begin{aligned} \mathcal{T}_T(f(t_1, \dots, t_n)) &= w_{T \cap \text{bind} f} f_{T \cup \text{bind}(f)} (\mathcal{T}_{T \cup \text{bind}(f)}(t_1), \dots, \mathcal{T}_{T \cup \text{bind}(f)}(t_n)) \\ \mathcal{T}_T(\pi X) &= \pi_T w_{T \cap \text{fresh}(X)} X_{T \setminus \text{fresh}(X)} \end{aligned} \quad (43)$$

The permutation  $\pi$  has its support included in  $T$  and  $\pi_T$  is the restriction of  $\pi$  to  $T$ . As in the previous section we can prove that all instances of a variable  $X$  have the same sort in the uniform equation, namely  $(T_E \cup \text{anc}(X)) \setminus \text{fresh}(X)$ . The proof of this is analogous to that of Lemma 6.6. The only difference is that now instead of reasoning only about abstractions  $[a]$ , we allow more general operation symbols. Similarly to Lemma 6.7, we get that  $\text{Fr}_{T(E)}(X) = \text{fresh}(X)$ .

**Example 6.13.** The  $\eta$  rule of the NEL theory for  $\alpha\beta\eta$ -equivalence of untyped  $\lambda$ -terms (Clouston and Pitts, 2007, Example 6.2)

$$a \not\# x \vdash L_a(A x V_a) \approx x$$

translates to

$$[a](\text{app}(w_a X, a)) = X : \emptyset$$

**Translation of semantics.** Consider a NEL theory as in (41) for a signature  $\text{Op}$ . Let  $\mathbb{X}$  be an algebra for this theory, that is, a nominal set  $|\mathbb{X}|$ , equipped with equivariant functions  $\text{Op}_n \times |\mathbb{X}|^n \rightarrow |\mathbb{X}|$ , for all arities  $n$ . ( $\text{Op}_n$  is the set of operation symbols of arity  $n$ , and is a nominal subset of  $\text{Op}$ .) We construct a sheaf  $\mathfrak{X}$  from  $|\mathbb{X}|$  as in (38). The sheaf algebra corresponding to  $\mathbb{X}$  is an algebra for the functor  $L$  with a uniform presentation obtained from  $\text{Op}$ , see the translation of syntax above. This sheaf algebra  $L\mathfrak{X} \rightarrow \mathfrak{X}$  maps

$$\{f_T(x_1, \dots, x_n)\}_{\mathfrak{X}} \mapsto \mathbb{X}[f](x_1, \dots, x_n)$$

where  $\mathbb{X}[f]$  is as in (Clouston and Pitts, 2007). This map is well-defined because of the equivariance of the functions  $\text{Op}_n \times |\mathbb{X}|^n \rightarrow |\mathbb{X}|$ .

**Theorem 6.14.** [semantic invariance] A structure  $\mathbb{X}$  for a NEL signature is a structure for a NEL theory as in (41) if and only if the sheaf algebra  $\mathfrak{X}$  obtained as above is an algebra for the translated uniform theory.

*Proof.* First we can check that a structure  $\mathbb{X}$  for a nominal signature satisfies a judgement  $\Delta \vdash t \approx t'$  iff the sheaf algebra  $\mathfrak{X}$ , constructed as above, satisfies the uniform equation  $E : T_E$  obtained as the translation of  $\Delta \vdash t \approx t'$ . The proof for this follows the same lines as the proof for Theorem 6.8. From a valuation  $\zeta'$  in  $\mathbb{X}$  of the variables in the freshness environment  $\Delta$  we get a valuation  $\zeta$  in  $\mathfrak{X}$  of the variables in some  $tr_S(E)$ , for  $S \cap T_E = \emptyset$ , and vice-versa. It remains to check that we have  $\llbracket t \rrbracket_{\zeta'}^{\mathbb{X}} = \llbracket \mathcal{T}t \rrbracket_{\zeta}^{\mathfrak{X}}$ , and this goes by induction on the structure of the terms. The next equalities hold in the underlying nominal set of  $\mathbb{X}$ .

$$\begin{aligned}
\llbracket \mathcal{T}(f(t_1, \dots, t_n)) \rrbracket_{\zeta}^{\mathfrak{X}} &= \llbracket w_{T \cap \text{bind} f} f_{T \cup \text{bind}(f)}(\mathcal{T}_{T \cup \text{bind}(f)}(t_1), \dots, \mathcal{T}_{T \cup \text{bind}(f)}(t_n)) \rrbracket_{\zeta}^{\mathfrak{X}} \\
&= \mathfrak{X}(w_{T \cap \text{bind} f})(\llbracket f_{T \cup \text{bind}(f)}(\mathcal{T}_{T \cup \text{bind}(f)}(t_1), \dots, \mathcal{T}_{T \cup \text{bind}(f)}(t_n)) \rrbracket_{\zeta}^{\mathfrak{X}}) \\
&= \llbracket f_{T \cup \text{bind}(f)}(\mathcal{T}_{T \cup \text{bind}(f)}(t_1), \dots, \mathcal{T}_{T \cup \text{bind}(f)}(t_n)) \rrbracket_{\zeta}^{\mathfrak{X}} \\
&= \{f_{T \cup \text{bind}(f)}(\llbracket t_1 \rrbracket_{\zeta}^{\mathfrak{X}}, \dots, \llbracket t_n \rrbracket_{\zeta}^{\mathfrak{X}})\}_{\mathfrak{X}} \\
&= \{f_{T \cup \text{bind}(f)}(\llbracket t_1 \rrbracket_{\zeta'}^{\mathbb{X}}, \dots, \llbracket t_n \rrbracket_{\zeta'}^{\mathbb{X}})\}_{\mathfrak{X}} \\
&= \mathbb{X}\llbracket f \rrbracket(\llbracket t_1 \rrbracket_{\zeta'}^{\mathbb{X}}, \dots, \llbracket t_n \rrbracket_{\zeta'}^{\mathbb{X}}) \\
&= \llbracket f(t_1, \dots, t_n) \rrbracket_{\zeta'}^{\mathbb{X}}
\end{aligned}$$

For axioms of the form  $X_1, \dots, X_n \vdash a \not\# f(X_1, \dots, X_n)$ , semantical invariance follows since the operation symbol  $f$  corresponds on the side of uniform signatures to operation symbols whose arities have the property that  $a$  does not belong to the result. So, for any valuation of the variables  $X_i$  in  $\mathfrak{X}$ , a translation of  $f(X_1, \dots, X_n)$  is evaluated to an element  $y$  of  $\mathfrak{X}(S)$  for a finite set  $S$ , with  $a \notin S$ . This means that, if  $\mathfrak{X}$  comes from a nominal set  $|\mathbb{X}|$ , we have that  $a$  is fresh for  $y$  in  $|\mathbb{X}|$ .  $\square$

**Corollary 6.15.** Theorem 5.23 and Theorem 6.14 give an HSPA theorem for models of NEL.

## 7. Conclusions

We studied universal algebra over nominal sets based on the observation that the category  $\mathbf{Nom}$  of nominal sets is a full reflective subcategory of an equationally definable class of many-sorted algebras, namely the presheaf category  $\mathbf{Set}^{\mathbb{I}}$ . As an application we proved two versions of Birkhoff's HSP theorem over nominal sets.

Section 3 took a category theoretic approach and investigated how to push the standard category theoretic proof of Birkhoff's HSP theorem through the adjunction relating  $\mathbf{Nom}$  and  $\mathbf{Set}^{\mathbb{I}}$ . Theorem 3.5 summarises the category theoretic assumptions needed to prove such an HSP theorem and Theorem 3.12 specialises it to nominal sets.

Section 5 is based on the particular structure of  $\mathbf{Set}^{\mathbb{I}}$  and introduced the notion of a uniform equational theory: uniform equations are invariant under shifting to larger contexts. This notion enabled us to extend Theorem 3.12 to prove a universal-algebra analogue (Theorem 5.18) of Gabbay's HSPA theorem (Gabbay, 2008).

Finally, Section 6 showed that, moving from  $\mathbf{Nom}$  to  $\mathbf{Set}^{\mathbb{I}}$ , the theories of Gabbay and Mathijssen (Gabbay and Mathijssen, 2009) and Clouston and Pitts (Clouston and Pitts, 2007) translate into uniform equational theories. This gives a new way of comparing the two different approaches and also new proofs of HSPA theorems for nominal algebra (Gabbay, 2008) and nominal equational logic (Clouston and Pitts, 2007).

There is a range of possibilities for further development. Obviously, more results of universal algebra should be made available for algebras over nominal sets. Related to this, our approach could be useful in ‘nominalising’ other areas of theoretical computer science based on universal algebra such as the theory of automata and formal languages. Another possibility is to profit from the fact that, unlike in the work of (Gabbay and Mathijssen, 2009; Clouston and Pitts, 2007), on the universal algebra side we are not forced to remain inside the uniform fragment; this could be of interest to, e.g., the investigation of bounded variable fragments of  $\lambda$ -calculus or first-order logic. It would also be interesting to consider an enriched categorical perspective, in particular to investigate connections between our many sorted equational logic and the synthetic nominal equational logic of (Fiore and Hur, 2008). Finally, one motivation for our study was to give a foundation for the work on logics for nominal calculi in the style of (Bonsangue and Kurz, 2007), where much still remains to be explored. For example, the models for the  $\pi$ -calculus of (Stark, 2008) should fit in the realm of universal algebra over nominal sets.

## References

- Adámek, J., Herrlich, H., and Strecker, G. E. (1990). *Abstract and Concrete Categories*. John Wiley & Sons.
- Adámek, J. and Rosický, J. (1994). *Locally Presentable and Accessible Categories*. CUP.
- Adámek, J., Rosický, J., and Vitale, E. M. *Algebraic Theories: a Categorical Introduction to General Algebra*. Draft available at <http://www.iti.cs.tu-bs.de/~adamek/algebraic.theories.pdf>.
- Bonsangue, M. and Kurz, A. (2007). Pi-calculus in logical form. In *LICS*.
- Bonsangue, M. M. and Kurz, A. (2006). Presenting functors by operations and equations. In *FoSSaCS*, volume 3921 of *LNCS*. Springer.
- Borceux, F. (1994). *Handbook of Categorical Algebra*. Cambridge University Press.
- Clouston, R. and Pitts, A. (2007). Nominal equational logic. In *Computation, Meaning and Logic, Articles dedicated to Gordon Plotkin*, volume 172 of *Electronic Notes in Theoretical Computer Science*. Elsevier.
- Fiore, M., Plotkin, G., and Turi, D. (1999). Abstract syntax and variable binding. In *LICS*.
- Fiore, M. P. and Hur, C.-K. (2008). Term equational systems and logics: (extended abstract). *Electr. Notes Theor. Comput. Sci.*, 218:171–192.
- Gabbay, M. (2008). Nominal algebra and the HSP theorem. *J. Logic Computation*. doi:10.1093/logcom/exn055.
- Gabbay, M. and Pitts, A. (1999). A new approach to abstract syntax involving binders. In *LICS*.
- Gabbay, M. J. and Mathijssen, A. (2009). Nominal (universal) algebra: equational logic with names and binding. *Journal of Logic and Computation*. In press.
- Johnstone, P. T. (2002). *Sketches of an Elephant: A Topos Theory Compendium, vol. 1 (Oxford Logic Guides, 43)*. Oxford University Press, USA.

- Kurz, A. and Petrişan, D. (2008). Functorial coalgebraic logic: The case of many-sorted varieties. In *CMCS'08*.
- Kurz, A. and Rosický, J. (July 2006). Strongly complete logics for coalgebras. Submitted.
- Maclane, S. and Moerdijk, I. (1994). *Sheaves in Geometry and Logic : A First Introduction to Topos Theory (Universitext)*. Springer.
- Stark, I. (2008). Free-algebra models for the pi -calculus. *Theor. Comput. Sci.*, 390(2-3):248–270.