

Relation lifting, A survey

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Abstract

We survey work in category theory and coalgebra on how to extend a functor from maps to relations. This relation lifting has a universal property, which is presented in some detail and guides us to generalisations to monotone and many-valued relations. As applications, it is shown how different notions of bisimulation, simulation and modal logics do arise.

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1. Introduction

The former RelMiCS and now RAMiCS conference series, established in 1994, is witness to the importance of relational methods in Computer Science. The change of the name to RAMiCS emphasises the close relationship between relational methods and algebraic methods. But what then about coalgebraic methods?

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This question seems even more appropriate if one remembers that relational methods play an important role in coalgebra. Apart from the fact that coalgebras themselves generalise relational structures such as Kripke frames, central coalgebraic notions such as (bi)simulation and Moss’s modal operator ∇ depend on relational methods. This paper presents an attempt to explain to researchers who work with relational methods, how and why they are used in coalgebra.

Of course, bisimulation and modal logic predate coalgebras by many decades. We therefore spend some time in Section 2 to present the specific coalgebraic perspective. While the examples for coalgebras (infinite trees, automata, Kripke models, Markov chains) are all familiar, the coalgebraic challenge is to develop a theory that *uniformly* applies to all of these, and many more, examples. The benefit then is that these coalgebraic methods can be extended modularly to combinations of the basic examples by arbitrary combinations of type constructors such as composition, product, coproduct etc.

To achieve this uniformity, coalgebra is developed parametrically in a type-functor T . In this paper, we employ from category theory a method to lift a relation on X and Y

$$X \multimap \rightarrow Y$$

to a relation on TX and TY

$$TX \multimap \rightarrow TY$$

and Section 2 finishes by giving an elementary description of it.

Next to the uniformity of coalgebra in the type-functor T , another thread through this paper is exploring some of the *generalisations* suggested by the category theoretic treatment. Thus, in Section 3, after describing the universal property of the relation lifting, we show that it generalises, with some technical effort but conceptually smoothly, to, first, monotone relations and, then, also to many-valued relations (of which metrics are a typical example).

Section 4 then presents the motivating examples, (bi)simulation and coalgebraic logic. We hope that on a first reading Section 4 is informative even if jumping over the more technical Section 3.

Finally, we conclude with some directions of future research.

Remark on notation. There exist conflicting traditions in which order to write relational composition. We apologise for writing it the ‘wrong way around’ (if compared to, eg, [48, 47]). The reason is that a central role in these notes is played by the functor $(-)_* : \text{Set} \rightarrow \text{Rel}$ which takes a function f to its graph relation f_* and we want the equality

$$(f \circ g)_* = f_* \cdot g_*$$

where \circ is composition of functions and \cdot is composition of relations. Since the order of composition of functions seems to be too well-established, we make composition of relations follow the composition of functions. One may think that it is unfortunate that the convention of writing the composition of functions diagrammatically/relationally, employed in famous category theoretical literature such as [38, 13], did not catch on.

2. Algebras, Coalgebras, Relations, Relation Lifting

We explain *algebras* and *coalgebras* for a functor and *relation lifting*.

1. The algebras for a functor are a (rather mild) generalisation of *algebras for a signature* known from universal algebra. Therefore, in this section, we spend some time on the move from signatures to general (finitary) functors. Algebras for a general functor then encode algebras for a signature that *satisfy certain equations*.

Hence the study of algebras for a functor (and their homomorphisms) does not really go beyond the realm of “classical” *universal algebra*.

2. Coalgebras for a functor are formally defined dually to algebras. However, one cannot quite say that the study of coalgebras (often called *universal coalgebra*) is the “dual of universal algebra”. The reason is that coalgebras can be used to encode various types of *automata*. Hence the questions one can naturally ask about coalgebras have a distinctive *computational meaning*.
3. One particular topic that coalgebras allow us to study is *behavioural equivalence of automata* and *bisimulations*. As it turns out, it is possible to develop such a theory *parametrically in the ambient functor* that describes the “behaviour” of automata. A crucial part of such theory is the notion of *relation lifting*: the possibility to extend a relation between X and Y to a relation between TX and TY , where T is the functor describing the automata.

2.A. Algebras for a functor

In universal algebra, to specify a class of algebras one starts with a signature $\Sigma : \mathbb{N} \rightarrow \mathbf{Set}$, or, equivalently, with a polynomial functor $F_\Sigma(X) = \coprod_{n \in \mathbb{N}} \Sigma(n) \bullet X^n$, where $\Sigma(n) \bullet X^n$ denotes the $\Sigma(n)$ -fold coproduct of the set X^n . To regard a signature Σ as a functor $F_\Sigma : \mathbf{Set} \rightarrow \mathbf{Set}$ allows us to say that an algebra is simply an arrow

$$F_\Sigma(X) \longrightarrow X$$

in the category \mathbf{Set} and that an algebra homomorphism $f : X \rightarrow X'$ is a commuting square

$$\begin{array}{ccc} F_\Sigma(X) & \longrightarrow & X \\ F_\Sigma(f) \downarrow & & \downarrow f \\ F_\Sigma(X') & \longrightarrow & X' \end{array} \quad (1)$$

Example 2.1. Suppose we specify a signature consisting of two binary operations $*$ and $+$ and one nullary operation e . Thus, the corresponding $\Sigma : \mathbb{N} \rightarrow \mathbf{Set}$ is defined by putting $\Sigma(2) = \{*, +\}$, $\Sigma(0) = \{e\}$ and $\Sigma(n) = \emptyset$ otherwise. The appropriate polynomial endofunctor $F_\Sigma : \mathbf{Set} \rightarrow \mathbf{Set}$ then collapses to

$$F_\Sigma(X) = \{e\} \bullet X^0 + \{*, +\} \bullet X^2$$

since the signature Σ is empty for $n \notin \{0, 2\}$. A typical element of $F_\Sigma(X)$ therefore can be conceived as having one of the following three forms

$$\begin{array}{ccc} e & * & + \\ \downarrow & \swarrow \searrow & \swarrow \searrow \\ 0 & x' \quad x & x' \quad x \end{array}$$

where we denoted by 0 the unique element of the set X^0 and (x', x) denotes an arbitrary element of X^2 .

Thus, elements of $F_\Sigma(X)$ are precisely the *flat terms* in variables X for the signature Σ . A mapping $a : F_\Sigma(X) \rightarrow X$ that makes X into an algebra for F_Σ is then simply the *interpretation* of flat terms in X . Thus, the mapping a sends the above three typical elements to their “meanings” in X .

It is now straightforward to verify that the commutative square (1) encodes precisely the fact that the mapping $f : X \rightarrow X'$ respects the operations e , $*$ and $+$.

In category theory, the notion of algebra for a signature is generalised to the notion of an algebra for a functor. Looking at (1) above, we see that it makes sense to speak of algebras $F_X \rightarrow X$ and their homomorphisms whenever we have a functor $F : \mathcal{C} \rightarrow \mathcal{C}$ on an arbitrary category \mathcal{C} .

What is gained by this generalisation?

Answer 1. Maybe not too much, as long as one stays in sets, that is, as long as one takes $\mathcal{C} = \mathbf{Set}$. Let us call a functor $\mathbf{Set} \rightarrow \mathbf{Set}$ *finitary* if it is fully determined by its action on finite sets. Without going into the category theoretic definition of finitary, it suffices to say here that an arbitrary functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ is finitary iff there is a signature Σ such that F is a quotient of some F_Σ ,

$$F_\Sigma \twoheadrightarrow F .$$

It follows that for any finitary $F : \mathbf{Set} \rightarrow \mathbf{Set}$, an F -algebra $F X \rightarrow X$ is nothing but an algebra

$$F_\Sigma X \twoheadrightarrow F X \longrightarrow X$$

for the signature Σ (and the *equations* defining the quotient $F_\Sigma \twoheadrightarrow F$). To summarize, the study of algebras for (finitary) functors $\mathbf{Set} \rightarrow \mathbf{Set}$ does not lead beyond the study of varieties in universal algebra. For a detailed account on algebras for a functor see [7].

Answer 2. Quite a lot is gained when moving to other categories \mathcal{C} than \mathbf{Set} . Ever since the work of Scott and others on domain theory and program semantics, type constructors T have been viewed as functors and semantic domains as (particular) algebras $T X \rightarrow X$, see e.g. [2]. Typically, the category \mathcal{C} is a category of partial orders or metric spaces, possibly with some completeness requirements.

Another interesting choice for \mathcal{C} is the category which is *dual* to \mathbf{Set} . One then obtains the notion of a *coalgebra* for a functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ as a function

$$X \longrightarrow T X .$$

As opposed to what we have seen in Answer 1 above, the fact that a finitary T is a quotient $F_\Sigma \twoheadrightarrow T$ of a polynomial functor F_Σ does not allow us to reduce the notion of a T -coalgebra to the notion of a coalgebra $X \rightarrow F_\Sigma X$ for a signature Σ . Going beyond polynomial functors will lead to new and interesting examples, as we are going to see next.

2.B. Coalgebras for a functor

Examples of coalgebras below show that coalgebras for polynomial functors F_Σ are of interest, but also that new phenomena such as *bisimulation* come into focus when going beyond polynomial functors.

Example 2.2. *Coalgebras for polynomial functors describe infinite trees.* For example, an element x in a coalgebra $\xi : X \rightarrow X + X$ can be seen as an infinite stream of left/right decisions: in state x , taking a transition by applying ξ yields a successor state $\xi(x)$ in either the left or the right component of $T X = X + X$.

Similarly, a state in a coalgebra $X \rightarrow A + B \times X \times X$ represents a possibly infinite tree with leaves labelled by elements of A and non-leaf nodes labelled by elements of B . Here, the polynomial functor is $T X = A + B \times X \times X$.

Example 2.3. *Coalgebras for the powerset functor are transition systems.* Here the functor T assigns the powerset $P X$ to every set X . Thus a coalgebra $\xi : X \rightarrow T X$ can be seen as describing the behaviour of a nondeterministic transition system: the “next state” $\xi(x)$ of a state x is, in fact, the subset $\xi(x) \subseteq X$ of all possible states into which x can evolve.

Example 2.4. *Coalgebras for the distribution functor are probabilistic transition systems.* Denote by $D X$ the set of all functions $p : X \rightarrow [0; 1]$ that have a finite support (i.e., such that $p(x) = 0$ for all but finitely many $x \in X$) and that satisfy $\sum_{x \in X} p(x) = 1$. Then a coalgebra $\xi : X \rightarrow D X$ describes a transition system with $\xi(x) : X \rightarrow [0; 1]$ giving the probability $\xi(x)(x')$ that x evolves to x' .

In universal coalgebra, a notion coined by Rutten in [45], therefore, it is important to develop the theory of T -coalgebras parametric in a functor T , much in the same way as universal algebra is done parametrically in a signature Σ . Some questions that arise are the following.

- For which functors $T : \mathbf{Set} \rightarrow \mathbf{Set}$ is there a final coalgebra?
- Can the behavioural equivalence given by the final coalgebra be characterised in terms of bisimulations?
- In universal algebra every signature Σ gives rise to an equational logic. Can we associate a coalgebraic logic to every functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$?
- How much of this can be done axiomatically, replacing \mathbf{Set} by general categories \mathcal{C} ?

Pioneering work on the first three items has been done, respectively, by [3, 45, 40], but the last item is still a topic of research. Of course, there are many further topics in coalgebra, for example, the use of coalgebra to solve recursive equations or to describe and derive congruence formats of process algebras or to extend and apply coalgebraic logic to description logics and knowledge representation. The topic of the present survey is a particular technique that helps to answer the questions above, namely, the *relation lifting*.

2.C. Relations, categorically

What are relations from a categorical view point? Let us begin with the obvious, namely that we have a category \mathbf{Rel} of sets and relations. A relation

$$R : A \multimap B$$

determines, and is determined by, its graph $\mathcal{G}R \subseteq A \times B$ and the domain projection $dR : \mathcal{G}R \rightarrow A$ and the codomain projection $cR : \mathcal{G}R \rightarrow B$. Given relations

$$A \multimap B \quad \text{and} \quad B \multimap C$$

there is the composition

$$A \multimap C$$

which is defined by the graph

$$\mathcal{G}(S \cdot R) = \{(a, c) \mid \exists b \in B. (a, b) \in R \wedge (b, c) \in S\}. \quad (2)$$

As we are following a methodology in which data types form categories and type constructors are functors acting on maps between data types, we seek a view of relations that allows us to extend map-based notions to relations.

First, to see how \mathbf{Set} sits inside \mathbf{Rel} , note that every map $f : A \rightarrow B$ gives rise to two relations. The first is written as

$$A \multimap B$$

and has the graph $\{(a, f(a)) \mid a \in A\}$. The second is its converse, written as

$$B \multimap A$$

and has the graph $\{(f(a), a) \mid a \in A\}$. Moreover, as it is well-known [48, Def 4.2.1], the relations of the form f_* or f^* for some map f can be recognised inside \mathbf{Rel} as precisely the left adjoints:

Proposition 2.5. *A relation R is of the form f_* for some map f iff R is a left adjoint in \mathbf{Rel} , that is, iff there is a (necessarily unique) relation S such that*

$$\text{Id} \subseteq S \cdot R \quad R \cdot S \subseteq \text{Id}$$

Moreover, $S = f^*$.

Second, we note that every relation $R : A \multimap B$ can be *tabulated* as a span of maps

$$\begin{array}{ccc} & \mathcal{G}R & \\ dR \swarrow & & \searrow cR \\ A & & B \end{array}$$

Note that such a span is *mono* in the sense that if $dR \circ f = dR \circ g$ and $cR \circ f = cR \circ g$ then $f = g$. Importantly, one can recover the relation from the maps:

Proposition 2.6. $R = cR_* \cdot dR^*$.

Let us note in passing that *any* span (W, f, g) represents a relation, namely the relation $g_* \cdot f^*$, which has the graph $\{(x, y) \mid \exists w \in W . f(w) = x \ \& \ g(w) = y\}$. Typically, it is the case that the original span (W, f, g) need not be of the form $(\mathcal{G}R, cR, dR)$ for $R = g_* \cdot f^*$. We will see in the next subsection how spans representing the same relation can be characterised.

As important special cases we note that for every map $f : A \rightarrow B$, the relations f_* and f^* are tabulated by spans of the form

$$\begin{array}{ccc} & A & \\ \text{id} \swarrow & & \searrow f \\ A & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & A & \\ f \swarrow & & \searrow \text{id} \\ B & & A \end{array} \quad (3)$$

respectively.

To summarize, relations can be tabulated as spans. That spans are pairs of arrows to which we can apply functors suggests how to define relation lifting.

2.D. Relation lifting

Relation lifting via spans. Given an arbitrary functor $T : \text{Set} \rightarrow \text{Set}$, we want to lift a relation

$$R : A \multimap B$$

to a relation

$$\bar{T}R : TA \multimap TB .$$

According to the previous paragraph, we should tabulate R as $(\mathcal{G}R, dR, cR)$, then apply T to dR and cR as in

$$\begin{array}{ccc} & T(\mathcal{G}R) & \\ T(dR) \swarrow & & \searrow T(cR) \\ TA & & TB \end{array}$$

and then reconstruct a relation using $(-)_*$ and $(-)^*$ in order to obtain

$$\bar{T}R = T(cR)_* \cdot T(dR)^* \quad (4)$$

Observe that the above definition entails equalities $\bar{T}(f_*) = (Tf)_*$ and $\bar{T}(f^*) = (Tf)^*$, if we use tabulations (3) above. Thus, the relation lifting via spans commutes with taking both graph formations f_* and f^* .

Explicitly, according to the definition of composition and of $(-)_*$ and $(-)^*$, the definition of relation lifting in (4) amounts to

$$\bar{T}R = \{(t, s) \in TA \times TB \mid \exists w \in T\mathcal{G}R . T\pi_1(w) = t, T\pi_2(w) = s\}, \quad (5)$$

which we will explicate in a number of examples:

Example 2.7. *Polynomial functors.* In the example of A -labelled binary trees, where $TX = 1 + A \times X \times X$, we simply have

$$(a, x_1, x_2) \bar{T}R(b, y_1, y_2) \iff a = b \ \& \ x_1 R x_2 \ \& \ y_1 R y_2$$

Example 2.8. *Powerset functor.* Let $R \subseteq X \times Y$ and $a \subseteq X, b \subseteq Y$. Then that the two subsets a, b are related by $\bar{\mathcal{P}}(R)$ amounts to

$$a(\bar{\mathcal{P}}R)b \iff \exists w \in \mathcal{P}(\mathcal{G}R) . \pi_1[w] = a \ \& \ \pi_2[w] = b,$$

where square brackets denote direct image. But this condition is equivalent to

$$a(\bar{\mathcal{P}}R)b \iff (\forall x \in a . \exists y \in b . x R y) \ \& \\ (\forall y \in b . \exists x \in a . x R y)$$

Example 2.9. *Distribution functor.* Recall from Example 2.4 that DX denotes the set of all probabilistic distributions on X , i.e., functions $p : X \rightarrow [0; 1]$ with finite support such that $\sum_{x \in X} p(x) = 1$. Given a mapping $f : X \rightarrow Y$, then Df sends a probabilistic distribution p on X to the probabilistic distribution $(Df)(p) : Y \rightarrow [0; 1]$ with $(Df)(p)(y) = \sum_{\{x | f(x)=y\}} p(x)$. The formula (5) then instantiates to

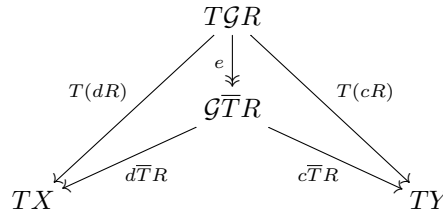
$$p \bar{D} q \iff \exists w \in D(\mathcal{G}R) . (D\pi_1)(w) = p \ \& \ (D\pi_2)(w) = q$$

Hence probability distributions p, q on X and Y , respectively, are related by \bar{D} iff there is a probability distribution w on the set $\{(x, y) | (x, y) \in R\}$ such that the equations

$$p(x) = \sum_{\{y | (x, y) \in R\}} w(x, y) \quad \text{and} \quad q(y) = \sum_{\{x | (x, y) \in R\}} w(x, y)$$

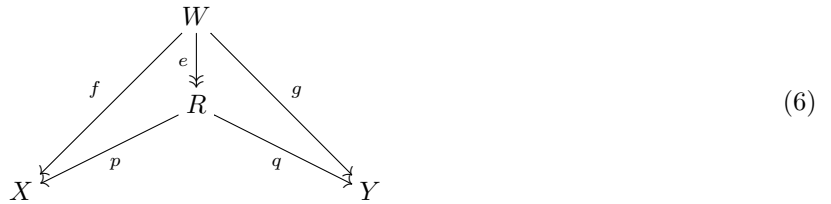
hold for every x in X and every y in Y .

Relation lifting does not need to preserve graphs. In Example 2.7 of a polynomial functor, the relation lifting was simple because in this case lifting preserves graphs of relations, that is, because for a polynomial T we have $\mathcal{G}(\bar{T}R) \cong T(\mathcal{G}R)$. This is not true in the other two Examples 2.8 and 2.9. Nevertheless, the graph $\mathcal{G}\bar{T}R$ of $\bar{T}R$ can be constructed from $T\mathcal{G}R$ by factoring the span $(T\mathcal{G}R, T(dR), T(cR))$ through a surjection e and a jointly injective span



so that $\mathcal{G}\bar{T}R$ is a subset of $TX \times TY$, namely the graph of $\bar{T}R$.

Relation lifting is independent of choice of span. The discussion above highlights that different spans may represent the same relation. We therefore would like to convince ourselves that the relation lifting does not depend on which particular spans are chosen as long as they represent the same relation. To this end let $q \circ e = g$ and $p \circ e = f$ in the diagram



It is immediate from the respective definitions that the two spans (W, f, g) and (R, p, q) describe the same relation iff $q_* \cdot p^* = g_* \cdot f^*$ if e is epi. Indeed, e epi is equivalent to $e_* \cdot e^* = \text{Id}$, so that we obtain $g_* \cdot f^* = q_* \cdot e_* \cdot e^* \cdot p^* = q_* \cdot p^*$. Since all functors $T : \text{Set} \rightarrow \text{Set}$ preserve epis, we have $(Te)_* \cdot (Te)^* = \text{Id}$, from which we conclude

$$\begin{aligned} (Tg)_* \cdot (Tf)^* &= (Tq \circ Te)_* \cdot (Tp \circ Te)^* \\ &= (Tq)_* \cdot (Te)_* \cdot (Te)^* \cdot (Tp)^* \\ &= (Tq)_* \cdot (Tp)^* \end{aligned}$$

This implies the following

Proposition 2.10. *Let (f, g) and (h, k) be two spans representing the same relation. Then they give rise to the same relation lifting, i.e., the equality $(Tg)_* \cdot (Tf)^* = (Th)_* \cdot (Tk)^*$ holds.*

Proof. If (f, g) and (h, k) represent the same relation R , that is, if $k_* \cdot h^* = g_* \cdot f^*$, then from both spans there is an epi onto the span $(\mathcal{G}R, dR, cR)$ tabulating the graph of R . Now we use twice the fact established above that any functor $T : \text{Set} \rightarrow \text{Set}$ preserves diagrams such as (6). \square

Exact squares and weak pullbacks. We will treat so-called exact squares in more detail later when they become important for monotone and many-valued relations. For now it is enough to say that a square

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ A & & B \\ f \searrow & & \swarrow g \\ & C & \end{array} \quad (7)$$

is exact, if the two relations defined by the span and the cospan agree, that is, if $q_* \cdot p^* = g_* \cdot f^*$.

Proposition 2.11. *Let (7) be a diagram in Set.*

$q_* \cdot p^* \subseteq g_* \cdot f^*$ iff (7) commutes.

$q_* \cdot p^* = g_* \cdot f^*$ iff (7) is a weak pullback.

Relation lifting \bar{T} preserves composition iff T preserves weak pullbacks. In order to understand whether relation lifting preserves composition, consider the following diagram (where we write $RS = R \cdot S$ for a more compact notation)

$$\begin{array}{ccccc} & & \mathcal{G}(RS) & & \\ & & \uparrow e & & \\ & & P & & \\ dP \swarrow & & & \searrow cP & \\ \mathcal{G}S & & & & \mathcal{G}R \\ dS \swarrow & & cS \searrow & & dR \swarrow \\ & & & & cR \searrow \\ & & & & \end{array} \quad (8)$$

In that diagram, the lower zig-zag

$$R \cdot S = (cR)_* \cdot (dR)^* \cdot (cS)_* \cdot (dS)^*$$

abbreviated to

$$RS = cR_* \cdot dR^* \cdot cS_* \cdot dS^*$$

is the composition of S and R and the upper outside span

$$(\mathcal{G}RS, dRS, cRS)$$

is its tabulation, which is formed by taking (P, dP, cP) to be the pullback of (cS, dR) and then factoring through an epi e and a mono span (dRS, cRS) . Note that it follows from Prop. 2.10 and Prop. 2.11 that the zig-zag at the bottom and the outer span at the top of (8) describe the same relation.

Applying the functor T to Diagram (8), one calculates as in [10]

$$\begin{aligned} \overline{T}(RS) &= T(cRS)_* \cdot T(dRS)^* && \text{def of } \overline{T} \\ &= (TcR \circ TcP)_* \cdot (TdS \circ TdP)^* && \text{Prop. 2.10} \\ &= TcR_* \cdot TcP_* \cdot TdP^* \cdot TdS^* && \text{functoriality of } (\cdot)_*, (\cdot)^* \\ &\subseteq TcR_* \cdot TdR^* \cdot TcS_* \cdot TdS^* && \text{Prop. 2.11} \\ &= (\overline{TR}) \cdot (\overline{TS}) && \text{def of } \overline{T} \end{aligned}$$

We have shown that $\overline{T}(R \cdot S) \subseteq (\overline{TR}) \cdot (\overline{TS})$. By Prop. 2.11 we can replace the \subseteq in the derivation above by equality, if T preserves weak pullbacks. In other words, \overline{T} preserves composition if T preserves weak pullbacks.

For the converse assume that (7) is a weak pullback. Then we have $q_* \cdot p^* = g^* \cdot f_*$ by Prop. 2.11. Since \overline{T} is a 2-functor (it preserves the order between relations), it also preserves adjoints, and it follows $Tq_* \cdot Tp^* = Tg^* \cdot Tf_*$, which in turn implies by Prop. 2.11 that the T -image of (7) is a weak pullback. To summarize,

Theorem 2.12. *The relation lifting \overline{T} satisfies $\overline{T}(R \cdot S) \subseteq (\overline{TR}) \cdot (\overline{TS})$. Moreover, $\overline{T}(R \cdot S) = (\overline{TR}) \cdot (\overline{TS})$ if and only if T preserves weak pullbacks.*

A more careful analysis of the above argument will show that Rel is universal wrt the properties of preserving maps, order, and composition and that the relation lifting \overline{T} is the unique extension of T from Set to Rel with these properties. This is the topic of the next section, together with generalisations to monotone and many-valued relations.

Remark 2.13 (Relation lifting of standard functors). A set-functor T is called standard if every natural transformation $C_{1,0} \rightarrow T$ can be uniquely extended to a natural transformation $C_1 \rightarrow T$. Here, $C_1 : \text{Set} \rightarrow \text{Set}$ is the functor, constant at one-element set, and $C_{1,0}$ differs from C_1 only by putting $C_{1,0}\emptyset = \emptyset$.

Standard functors have many pleasant properties, for example they preserve inclusion. For technical reasons it is often convenient, and possible without loss of generality, to restrict attention to standard functors. In fact, every functor $T : \text{Set} \rightarrow \text{Set}$ has a standard reflection $\alpha_T : T \rightarrow \widehat{T}$, i.e., \widehat{T} is a standard functor and α_T has the obvious universal property. For a summary of some useful properties of the relation lifting of standard functors we refer to Section 3 of [33].

References. The definition of relation lifting as well as the propositions and the proof of the theorem are taken from Barr's paper [10] on relational algebras, with the only exception that he does not explicitly state the 'only if' part of the theorem, which was given in Trnková [53]. Our present account also follows Carboni-Kelly-Wood [17, Sections 2.2, 4.3] and Hermida [22], who prove more general theorems.

3. The universal property of Rel

Relations can be structured in many different ways. For us, the following facts are the most important.

Relations form a category, that is they can be composed and identity relations are neutral elements of composition. Moreover, relations are ordered by inclusion and composition is monotone (preserves inclusion).

Category theoretically, this can be summed up by saying that Rel is a 2-category (2-cells are inclusions), or more specifically, by saying that Rel is a Pos -enriched category (the set $\text{Rel}(A, B)$ of relations from A to B is partially ordered by inclusion).

For our purposes, we do not need to know the general theory of 2-categories and enriched categories as the relevant notions simplify greatly in the special case of interest here: A Pos -enriched category, or simply a Pos -category, is a category where the homsets are equipped with a partial order and the composition is monotone. A 2-functor or Pos -functor between two Pos -categories is a functor that preserves the order on homsets (such functors are also called locally monotone).

One of several reasons to emphasise the 2-categorical or Pos -enriched viewpoint is that we are mainly interested in locally monotone functors. In particular, for us a relation lifting should always satisfy the implication $R \subseteq S \Rightarrow \overline{T}R \subseteq \overline{T}S$. One of the technical benefits of working with 2-functors is that they preserve adjointness. For example, it will be important that $f_* \dashv f^*$ is mapped to an adjunction $\overline{T}(f_*) \dashv \overline{T}(f^*)$, in other words, that relation lifting preserves maps. An additional benefit of the Pos -enriched situation is that the order on the homsets is anti-symmetric and therefore adjoints are not only determined up to isomorphism but up to equality.

The next bit of important structure are the faithful functors

$$(-)_* : \text{Set} \longrightarrow \text{Rel} \quad (-)^* : \text{Set} \longrightarrow \text{Rel}^{\text{op}}$$

that map a function f to the same function f_* seen as a relation, and its converse f^* .

As in any 2-category, we can speak of adjunctions. Here, this means that a relation R has a left-adjoint L if and only if

$$\text{Id} \subseteq R \cdot L \quad L \cdot R \subseteq \text{Id}$$

This happens if and only if L is the graph of a map, that is, if there is a function f such that $L = f_*$.

3.A. The universal property of $\text{Rel}(\text{Set})$

The next theorem, which can be seen as an important reformulation of Theorem 2.12, explains in which sense the category Rel is universal over Set . Intuitively, Rel arises from Set by freely adding adjoints to all maps and by turning epis into split epis.

Theorem 3.1. *The functor $(-)_* : \text{Set} \longrightarrow \text{Rel}$ has the following three properties:*

1. $(-)_*$ preserves maps, that is, every f_* has a right-adjoint (denoted f^*).
2. $q_* \cdot p^* = g^* \cdot f_*$ for all weak pullbacks

$$\begin{array}{ccc}
 & W & \\
 p \swarrow & & \searrow q \\
 A & & B \\
 f \searrow & & \swarrow g \\
 & C &
 \end{array} \tag{9}$$

3. $e_* \cdot e^* = \text{Id}$ for all epis e .

Moreover, the functor $(-)_*$ is universal w.r.t. these three properties in the following sense: if \mathcal{K} is any Pos -category to give a locally monotone functor $H : \text{Rel} \longrightarrow \mathcal{K}$ is the same as to give a functor $F : \text{Set} \longrightarrow \mathcal{K}$ with the following three properties:

1. Every Ff has a right adjoint, denoted by $(Ff)^r$.

2. $Fq \cdot (Fp)^r = (Fg)^r \cdot Ff$ for all weak pullbacks as in (9).
3. $Fe \cdot (Fe)^r = \text{Id}$ for all epis e .

Proof. The three properties of $(-)_*$ have been established in Propositions 2.5-2.11. Furthermore, given H , the functor $F = H \circ (-)_*$ satisfies the three properties, because H preserves adjoints by virtue of being a 2-functor. Conversely, given a functor F with the three properties, we let $H(f_*) = Ff$. This determines H on a relation R via $H(R) = H(cR_* \cdot dR^*) = H(cR_*) \cdot H(dR^*) = F(cR) \cdot F(dR)^r$, using that H preserves composition and adjoints. And, analogous to the reasoning following Diagram (8), one shows that H preserves composition (using that F satisfies properties 2 and 3). \square

As usual in category theory, an abstract categorical account does not only lead to clean and elegant proofs but also to generalisations. In fact, the theorem above is only a special case of a theorem by Carboni-Kelly-Wood in [17, Section 4.3] about relations in regular categories and of theorems by Hermida in [22] about bicategories of spans in a category with pullbacks. We will see related but different generalisations in Sections 3.C and 3.D.

We finish this section with a colax version of the theorem, also due to [17], where preservation of composition is weakened to preservation of composition up to inequality on homsets. The proof is essentially the same as the one of Theorem 3.1, once one notices that the properties 1-3 of H below guarantee that adjoints $h_* \dashv h^*$ of maps h are preserved by H . Also note that these properties guarantee that $H(R \cdot S) \leq HR \cdot HS$.

Theorem 3.2. *If \mathcal{K} is any Pos-category, then to give $H : \text{Rel} \rightarrow \mathcal{K}$ satisfying*

1. $H(\text{Id}) = \text{Id}$,
2. $H(f_* \cdot R) = H(f_*) \cdot HR$,
 $H(R \cdot g^*) = H(R) \cdot H(g^*)$,
 $H(g^* \cdot f_*) \leq H(g^*) \cdot H(f_*)$,
3. $R \subseteq S \Rightarrow HR \leq HS$,

is the same as to give a functor $F : \text{Set} \rightarrow \mathcal{K}$ with the following three properties:

1. Every Ff has a right adjoint, denoted by $(Ff)^r$.
2. $Fq \cdot (Fp)^r \leq (Fg)^r \cdot Ff$ for all commuting squares (9).
3. $Fe \cdot (Fe)^r = \text{Id}$ for all epis e .

Remark 3.3. In [17, Section 2.2], a *morphism* of categories of relations is an operation H satisfying (a) $H(\text{id}) = \text{id}$ and (b) $g \cdot R \leq S \cdot f \Rightarrow Hg \cdot HR \leq HS \cdot Hf$. It follows from (a) and (b) that H is locally monotone, preserves adjoints, and restricts to a functor on maps.

Recall that $H : \text{Rel} \rightarrow \text{Rel}$ is *colax* if $H(\text{Id}) \subseteq \text{Id}$ and $H(R \cdot S) \subseteq HR \cdot HS$ and $R \subseteq S \Rightarrow HR \subseteq HS$. Conditions 1-3 of the theorem are equivalent to saying that H is a colax morphism, that is, that H satisfies (a), (b) and $H(R \cdot S) \leq H(R) \cdot H(S)$. Finally, we note that condition 2 can be replaced by $H(f_*) \cdot HR \leq H(f_* \cdot R)$ and $H(R \cdot S) \leq HR \cdot HS$.

3.B. Relation lifting via Kleisli categories and distributive laws

[This subsection presents a digression from the main thread. It can be skipped. But we think that it is important to point out that relation lifting can, and often is, described as a Kleisli lifting. As we will see, this description carries over to the generalisations of the following subsections.]

So far, one of the main ideas was to look at relations as spans. Another important idea, which is equivalent but has different generalisations, is to think of a relation as a many-valued function, that is, as a function

$$X \longrightarrow \mathcal{P}Y$$

where $\mathcal{P}X$ is the set of subsets of X . To compose

$$X \xrightarrow{f} \mathcal{P}Y \quad Y \xrightarrow{g} \mathcal{P}Z$$

we note that we can lift g from elements of Y to a function g^\sharp on subsets of Y , which allows us to define $g \cdot f = g^\sharp \circ f$. Formally, we define the operation $(-)^{\sharp} : \text{Set}(Y, \mathcal{P}Z) \longrightarrow \text{Set}(\mathcal{P}Y, \mathcal{P}Z)$ via

$$g^\sharp(b) = \bigcup \{g(y) \mid y \in b\}.$$

The abstract structure of this example has a simple axiomatisation [38, Chapter 1.3 (Definition 3.2 and Exercise 12)]:

Given a category \mathcal{C} , let M be a map (which is not required to be functorial at this stage) from objects of \mathcal{C} to objects of \mathcal{C} (think of $M = \mathcal{P}$ and $\mathcal{C} = \text{Set}$). Let $\eta_X : X \longrightarrow MX$ be a collection of arrows in \mathcal{C} and let $(-)^{\sharp}_{Y,Z}$ be a collection of functions $\mathcal{C}(Y, MZ) \longrightarrow \mathcal{C}(MY, MZ)$. Then $(M, \eta, (-)^{\sharp})$ is called a **Kleisli triple** if

$$\begin{aligned} \eta^\sharp &= \text{id} \\ f^\sharp \circ \eta &= f \\ (g^\sharp \circ f)^\sharp &= g^\sharp \circ f^\sharp \end{aligned}$$

In fact, the above axiomatisation allows us to extend the assignment $X \mapsto MX$ uniquely to a functor $M : \mathcal{C} \longrightarrow \mathcal{C}$ such that the collection η_X becomes a natural transformation from Id to M , and, when one defines $\mu_X : MMX \longrightarrow MX$ by putting $\mu_X = (\text{id}_{MX})^\sharp$, then μ is a natural transformation from MM to M . The above axioms guarantee that (M, η, μ) is a monad. Conversely, every monad (M, η, μ) yields a Kleisli triple by defining $f^\sharp = \mu_Z \circ Mf : MY \longrightarrow MZ$ for $f : Y \longrightarrow MZ$. See [38] for details.

Coming back to the example above, the assignment $X \mapsto \mathcal{P}X$, together with the collection $\eta_X(x) = \{x\}$ of singleton maps and with

$$g^\sharp(b) = \bigcup \{g(y) \mid y \in b\}$$

is easily seen to form a Kleisli triple.

Kleisli triples gives rise to categories, the Kleisli categories, which tend to resemble categories of relations. Given a Kleisli triple $(M, \eta, (-)^{\sharp})$ on a category \mathcal{C} , the **Kleisli category** $Kl(M)$, see [31, 37], has the same objects as \mathcal{C} and arrows $X \rightarrow Y$ in $Kl(M)$ are arrows $X \longrightarrow MY$ in \mathcal{C} . The identity on X in $Kl(M)$ is given by η_X and the composition $g \cdot f$ in $Kl(M)$ is given by the composition $g^\sharp \circ f$ in \mathcal{C} . As to be expected for a ‘category of relations’ there is an identity-on-objects functor

$$(-)_\star : \mathcal{C} \longrightarrow Kl(M)$$

taking an arrow $f : X \longrightarrow Y$ to a ‘map’ $\eta_Y \circ f : X \rightarrow Y$. On the other hand, there need to be no analogue of the converse of a relation nor of the order between relations.

Relation lifting via distributive laws. We discussed previously how to lift a relation by applying a functor T to the span tabulating the relation. Here, the idea is that a relation lifting \tilde{T} takes a relation

$$X \longrightarrow \mathcal{P}Y$$

and applies the functor T to $X \longrightarrow \mathcal{P}Y$ and composes with a so-called distributive law $T\mathcal{P} \longrightarrow \mathcal{P}T$ to obtain

$$TX \longrightarrow T\mathcal{P}Y \longrightarrow \mathcal{P}TY,$$

which is a relation between TX and TY .

Beck's theorems on distributive laws. In a famous paper, Beck [13] shows (among other things) that there is a one-to-one correspondence between certain liftings of functors and certain natural transformations called distributive laws. In fact, the article opens with the observation that the well-known distributive law in rings of multiplication over addition corresponds to a natural transformation

$$MonAb \longrightarrow AbMon$$

between the monad Mon for monoids and the monad Ab for abelian groups. This natural transformation, then also called a distributive law, is exactly what is needed to show that $AbMon$ is itself a monad (the monad of rings) and it is also exactly what is needed to show that Ab lifts from a monad on sets to a monad on monoids. Our situation is a variant on the above where \mathcal{P} is a monad but T is just a functor and where the lifting is not to a category of algebras but to a Kleisli category. The theory that allows us to consider Beck's original theorem and all of its variants as special cases of the same construction is due to Street [51].

The variant relevant for us is the theorem that states that a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$ can be extended to a functor \tilde{T} making the square

$$\begin{array}{ccc} Kl(M) & \xrightarrow{\tilde{T}} & Kl(M) \\ (-)_* \uparrow & & \uparrow (-)_* \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array} \quad (10)$$

commutative if and only if there is a natural transformation, called a distributive law of T over M ,

$$\lambda : TM \longrightarrow MT$$

satisfying

$$\begin{aligned} \lambda \circ T\eta &= \eta T \\ \mu T \circ M\lambda \circ \lambda M &= \lambda \circ T\mu \end{aligned}$$

To give a specific example, take $\mathcal{C} = \mathbf{Set}$ and $M = \mathcal{P}$ and λ the distributive law of T over \mathcal{P} such that $\tilde{T} = \bar{T}$. This distributive law is obtained from applying \bar{T} to the elementship relation

$$\begin{aligned} \ni_X : \mathcal{P}X &\dashrightarrow X \\ \bar{T}(\ni_X) : T\mathcal{P}X &\dashrightarrow TX \\ \lambda : T\mathcal{P}X &\longrightarrow \mathcal{P}TX \end{aligned}$$

A good illustration of the use of this distributive law is Power and Turi [43], where it is used to obtain trace semantics from the coalgebraic bisimulation semantics. See also [21] for more information on this topic.

Question: In the case of \mathbf{Set} the existence of a distributive law is weaker than the existence of a relation lifting as \tilde{T} in (10) is not required to be locally monotone. Is there an example of a Kleisli lifting \tilde{T} that is not a relation lifting?

In the ordered and many-valued setting considered below, this problem disappears if one takes, as we do, the perspective of enriched category theory where all functors are locally monotone. Accordingly, relation lifting and Kleisli lifting (for a suitable power-functor) match up.

3.C. Monotone relations

After having considered relations $R : A \multimap B$ as spans

$$\begin{array}{ccc} & \mathcal{G}R & \\ dR \swarrow & & \searrow cR \\ A & & B \end{array}$$

and as Kleisli-arrows

$$A \longrightarrow \mathcal{P}B$$

we now bring into the picture relations as functions

$$X \times Y \longrightarrow 2,$$

or rather as monotone functions

$$X^{\text{op}} \times Y \longrightarrow 2$$

where X, Y are posets (or preorders). For example, if X is a set of states equipped with an order of approximation and Y is a set of propositions ordered by implication or derivability, then a monotone relation

$$\Vdash : X \multimap Y$$

is a monotone map

$$X^{\text{op}} \times Y \longrightarrow 2,$$

where monotonicity amounts to saying that \Vdash is *weakening-closed*, that is,

$$\frac{x' \leq x \quad x \Vdash y \quad y \leq y'}{x' \Vdash y'}$$

Weakening-closed relations are pervasive in logic and domain theory and provide an important motivation for monotone relations. For example, if $X = Y$, then the order on a distributive lattice (or Heyting or Boolean algebra) is a monotone relation. Similarly, the ‘ \vdash ’ of sequent calculi is typically a monotone relation on the preorder of propositions. Another example is given by proximities (see condition (3) in Definition 3 of Smyth [49]), which play an important role in domain theory [27, 26, 29].

In the remainder of this section, we quickly go through the material we have seen so far, but generalise it to the ordered setting, referring to [14] for full details.² Even though everything we have said generalises with the appropriate modifications, notice that the converse R^{op} of a relation $R : A \multimap B$, being a monotone function $A^{\text{op}} \times B \longrightarrow 2$, gives rise to a converse $R^{\text{op}} : B^{\text{op}} \multimap A^{\text{op}}$ between the opposite orders. In particular, we cannot compose a relation with its converse.

As in (2), given monotone relations

$$A \xrightarrow{R} B \quad \text{and} \quad B \xrightarrow{S} C$$

²This generalisation is not covered by [17] or [22]. In this context it is worth noting that Pos and Preord are not regular categories, which would require that the pullback of a coequalizer is a coequalizer. To see that this is not always the case,

$$\text{consider the coequalizer } \begin{array}{ccc} & \begin{array}{c} c \\ \vdots \\ b_2 \end{array} & \\ & \uparrow & \\ & \begin{array}{c} c \\ \vdots \\ b_1 \end{array} & \\ & \uparrow & \\ a & & \end{array} \longrightarrow \begin{array}{c} c \\ \vdots \\ b \\ \vdots \\ a \end{array} \quad \text{and pull it back along the inclusion } \begin{array}{ccc} & \begin{array}{c} c \\ \vdots \\ b \end{array} & \\ & \uparrow & \\ & \begin{array}{c} c \\ \vdots \\ a \end{array} & \\ & \uparrow & \\ a & & \end{array} .$$

the composition

$$A \xrightarrow{S \cdot R} C$$

is given by

$$S \cdot R(a, c) = \bigvee_b R(a, b) \wedge S(b, c) \quad (11)$$

where \bigvee and \wedge are taken in the lattice $\mathfrak{2}$ (and can be read as ‘there exists’ and ‘and’, respectively).

The identity relation on A , also called A , is given by the order on A and we write

$$\text{Id}(a, a') = A(a, a')$$

instead of $a \leq_A a'$.

What should be the graph of a monotone function $f : A \rightarrow B$? Here we have two (ultimately equivalent) choices corresponding to whether we generalise $fa = b$ to $fa \leq b$ or $b \leq fa$. We can take as the graph of f the set $\{(a, b) \mid b \leq fa\}$ which corresponds to the relation $B \dashrightarrow A$

$$\lambda a, b. B(b, fa) : B^{\text{op}} \times A \rightarrow \mathfrak{2}$$

or we can take $\{(a, b) \mid fa \leq b\}$ which corresponds to the relation $A \dashrightarrow B$

$$\lambda a, b. B(fa, b) : A^{\text{op}} \times B \rightarrow \mathfrak{2}$$

As a mapping of f , the first, $f \mapsto \lambda a, b. B(b, fa)$, is monotone but contravariant, and the second, $f \mapsto \lambda a, b. B(fa, b)$, is covariant but antitone. In [14] we used the first alternative, here we use the second, that is, we let $f_* = \lambda a, b. B(fa, b)$. We do not need to be disturbed by the antitonicity, which is familiar in the special case where f, g are constants: $f \leq g \Leftrightarrow \uparrow g \subseteq \uparrow f$. In other words, the choice we have is analogous to the choice of whether one represents an element of a poset by its principal downset or principal upset.

Thus, we have functors

$$\begin{aligned} (-)_* : \text{Pos} &\rightarrow \text{Rel}(\text{Pos})^{\text{co}} \\ A &\mapsto B \\ f : A \rightarrow B &\mapsto \lambda a, b. B(fa, b) \\ f \leq g &\mapsto g_* \leq f_* \end{aligned}$$

where the co indicates that only the order on the 2-cells gets reversed and

$$\begin{aligned} (-)^* : \text{Pos} &\rightarrow \text{Rel}(\text{Pos})^{\text{op}} \\ A &\mapsto B \\ f : A \rightarrow B &\mapsto \lambda a, b. B(b, fa) \\ f \leq g &\mapsto f^* \leq g^* \end{aligned}$$

where the op indicates that only the direction of the 1-cells gets reversed. As in the unordered case we have $f_* \dashv f^*$ in $\text{Rel}(\text{Pos})$ with unit

$$A(a, a') \leq f^* \cdot f_*(a, a') = \bigvee_b B(fa, b) \wedge B(b, fa')$$

and counit

$$\bigvee_a B(b, fa) \wedge B(fa, b') = f_* \cdot f^*(b, b') \leq B(b, b')$$

where again \leq, \vee, \wedge are taken in $\mathbf{2}$. As in Proposition 2.5, it remains true that a monotone relation arises from a monotone function iff it is left-adjoint.³ And, as in Proposition 2.6, every monotone relation R can be tabulated as a span $(\mathcal{G}R, dR, cR)$. The definition of the **relation lifting** of a relation R wrt a functor $T : \mathbf{Pos} \rightarrow \mathbf{Pos}$ (or a functor $T : \mathbf{Preord} \rightarrow \mathbf{Preord}$) is then defined in the same way as for ordinary relations in (4):

$$\overline{TR} = T(cR)_* \cdot T(dR)^* \quad (12)$$

Remark 3.4. In the discrete case, an arbitrary span is (isomorphic to) the graph $(\mathcal{G}R, dR, cR)$ of a relation R if and only if the span is a mono-span (that is, if the corresponding map into the product is mono). In the ordered case, one needs to add a condition making sure that the relation corresponding to the span is monotone. We say that $(W, d_0 : W \rightarrow X, d_1 : W \rightarrow Y)$ is a weakening-closed span if

$$\forall z' \in W. \forall x \in X. x \leq d_0(z') \Rightarrow \exists z \in W. z \leq z' \ \& \ d_0(z) = x \ \& \ d_1(z) = d_1(z') \quad (13)$$

$$\forall z \in W. \forall y' \in Y. d_1(z) \leq y' \Rightarrow \exists z' \in W. z \leq z' \ \& \ d_0(z') = d_0(z) \ \& \ d_1(z') = y' \quad (14)$$

which is best remembered by the pictures

$$\begin{array}{ccc} x & \cdots \cdots & y' \\ \downarrow & & \parallel \\ x' & \cdots \cdots & y' \end{array} \quad \begin{array}{ccc} x & \xrightarrow{z} & y \\ \parallel & & \downarrow \\ x & \cdots \cdots & y' \end{array}$$

There is a bijective correspondence between weakening-closed mono-spans in \mathbf{Pos} (taken up to isomorphism) and monotone relations in \mathbf{Pos} . This correspondence preserves compositions of relations, that is, taking the pullback of two spans and factoring it onto/embedding corresponds to the composition of the corresponding monotone relations. \square

Relation lifting is independent of choice of span. If two weakening-closed spans

$$\begin{array}{ccc} & W & \\ f \swarrow & & \searrow g \\ X & & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} & W' & \\ f' \swarrow & & \searrow g' \\ X & & Y \end{array}$$

represent the same monotone relation $X \multimap Y$, then the maps $\langle f, g \rangle$ and $\langle f', g' \rangle$ into $X \times Y$ have the same image (with the order on the image inherited from the order on $X \times Y$). In other words, factoring the spans onto/embedding results in isomorphic spans. This leads us to consider the following diagram with $q \circ e = g$ and $p \circ e = f$.

$$\begin{array}{ccc} & W & \\ & \downarrow e & \\ & R & \\ f \swarrow & & \searrow g \\ X & & Y \\ p \swarrow & & \searrow q \end{array} \quad (15)$$

³We want to show that $A(a, a') \leq \bigvee_b R(a, b) \wedge S(b, a')$ and $\bigvee_a S(b, a) \wedge R(a, b') \leq B(b, b')$ only if $R(a, b) = B(fa, b)$ for some $f : A \rightarrow B$. First consider the special case where A is the one element set. Then R is an upset, S is a downset, and the two inequalities ensure that there is $f \in B$ such that $S = \downarrow f$ and $R = \uparrow f$, or, in our notation, $S(b, a) = B(b, f)$ and $R(a, b) = B(f, b)$. In the general case, the same reasoning gives an fa for each $a \in A$ with $S(b, a) = B(b, fa)$ and $R(a, b) = B(fa, b)$.

Remark: In the discrete case the two inequalities can be understood as separately defining that R is total and that R is single-valued. In the ordered case, being single-valued is replaced by being a principal upset, but this needs both inequalities simultaneously.

The crucial property here is again that e satisfies

$$e_* \cdot e^* = \text{Id}$$

or, explicitly,

$$R(r, r') = \bigvee_w R(r, ew) \wedge R(ew, r').$$

If R is a poset, then this property says exactly that e is onto. If R is a preorder, then it says that e hits all equivalence classes of R . In [12] such an e is said to be ‘absolutely dense’. In **Set** all onto maps split, but this is not the case in **Pos** or **Preord**. On the other hand, as we have just said, we still have that e is onto in **Pos** iff e_* splits in $\text{Rel}(\text{Pos})$.

Now, the same reasoning as above for Proposition 2.10, shows that if two weakening closed spans represent the same monotone relation, then they give rise to the same relation lifting.

Proposition 3.5. *The two weakening closed spans in Diagram (15) represent the same relation, ie $g_* \cdot f^* = q_* \cdot p^*$, if $e_* \cdot e^* = \text{Id}$.*

Exact squares. Exact squares were introduced by Guitart [20]. Their importance for us is that they generalise weak pullbacks to the ordered setting. A square

$$\begin{array}{ccc}
 & W & \\
 p \swarrow & & \searrow q \\
 A & \leq & B \\
 f \searrow & & \swarrow g \\
 & C &
 \end{array} \tag{16}$$

in **Pos** or **Preord** (or, more generally, in any **Pos**-category) is called *exact*, if

$$\begin{aligned}
 & f \circ p \leq g \circ q \\
 & \forall a, b. (fa \leq gb \Rightarrow \exists w. a \leq pw \ \& \ qw \leq b)
 \end{aligned}$$

The next proposition characterises exact squares in **Pos** and **Preord** as those squares where the upper span and the lower cospan represent the same relation:

$$q_* \cdot p^* = g_* \cdot f^*.$$

(Recall that $f \leq g \Leftrightarrow g_* \leq f_*$.)

Proposition 3.6. *Let (16) be a diagram in **Pos** or **Preord**.*

$q_* \cdot p^* \leq g_* \cdot f^*$ iff (16) commutes laxly, that is, iff $f \circ p \leq g \circ q$.

$q_* \cdot p^* \geq g_* \cdot f^*$ iff $\forall a, b. (fa \leq gb \Rightarrow \exists w. a \leq pw \ \& \ qw \leq b)$.

Proof. The first statement follows from using once each of the following general facts about adjoints $l \dashv r$.

$$l \cdot u \leq v \Leftrightarrow u \leq r \cdot v \quad \text{and} \quad k \cdot r \leq h \Leftrightarrow k \leq h \cdot l$$

For the second statement we note that $g_* \cdot f^* \leq q_* \cdot p^*$ is by definition of $(-)_*$ and $(-)^*$ the same as

$$\bigvee_c A(fa, c) \wedge B(c, gb) \leq \bigvee_w A(a, pw) \wedge B(qw, b),$$

which is equivalent to

$$C(fa, gb) \leq \bigvee_w A(a, pw) \wedge B(qw, b), \tag{17}$$

which in turn is just a different notation for $fa \leq gb \Rightarrow \exists w. a \leq pw \ \& \ qw \leq b$. \square

Relation lifting \bar{T} preserves composition iff T preserves exact squares. Recall Diagram (8). We consider the same diagram, but now in Pos or Preord,

$$\begin{array}{c}
 \mathcal{G}(RS) \\
 \uparrow e \\
 P \\
 \swarrow dP \quad \searrow cP \\
 \mathcal{G}S \quad \le \quad \mathcal{G}R \\
 \swarrow dS \quad \searrow cS \quad \swarrow dR \quad \searrow cR \\
 \downarrow dRS \quad \downarrow cRS
 \end{array} \tag{18}$$

where now (P, dP, cP) is the the ordered version of a pullback, known as a comma-square, and defined as $P = \{(s, r) \mid cS(s) \leq dR(r)\}$. As before e is the surjection (epi) that arises from factoring the span $(dS \circ dP, cR \circ cP)$ through a mono-span (dRS, cRS) . Exactly as in the reasoning for Theorem 2.12 we obtain the analogous theorem for monotone relations.

Theorem 3.7. *Let T be an endofunctor on Pos preserving epis. Then the relation lifting \bar{T} satisfies the inclusion $\bar{T}(R \cdot S) \subseteq (\bar{T}R) \cdot (\bar{T}S)$. Moreover, $\bar{T}(R \cdot S) = (\bar{T}R) \cdot (\bar{T}S)$ if and only if T preserves exact squares.*

It is worth noting that preserving exact squares implies preserving epis.

Analogous to Theorem 3.1, one can show that the category $\text{Rel}(\text{Pos})$, and $\text{Rel}(\text{Preord})$, of monotone relations has a universal property.

Theorem 3.8. *The locally monotone functor $(-)_* : \text{Pos} \rightarrow \text{Rel}(\text{Pos})^{\text{co}}$ has the following three properties:*

1. $(-)_*$ preserves maps, that is, every f_* has a right-adjoint f^* in $\text{Rel}(\text{Pos})$.
2. $q_* \cdot p^* = g^* \cdot f_*$ for all exact squares

$$\begin{array}{c}
 W \\
 \swarrow p \quad \searrow q \\
 A \quad \le \quad B \\
 \swarrow f \quad \searrow g \\
 C
 \end{array} \tag{19}$$

3. $e_* \cdot e^* = \text{Id}$ for all epis e .

Moreover, the functor $(-)_*$ is universal w.r.t. these three properties in the following sense: if \mathcal{K} is any Pos-category to give a locally monotone functor $H : \text{Rel}(\text{Pos}) \rightarrow \mathcal{K}$ is the same as to give a locally monotone functor $F : \text{Pos} \rightarrow \mathcal{K}^{\text{co}}$ with the following three properties:

1. Every Ff has a right adjoint in \mathcal{K} , denoted by $(Ff)^r$.
2. $Fq \cdot (Fp)^r = (Fg)^r \cdot Ff$ for all exact squares as in (19).
3. $Fe \cdot (Fe)^r = \text{Id}$ for all epis e .

3.D. Many-valued relations

We now turn our attention to the relation lifting of many-valued relations. A *many-valued* relation is a generalisation of a monotone relation in the sense that in

$$R : A^{\text{op}} \times B \longrightarrow \mathbb{2}$$

one replaces the two-element lattice $\mathbb{2}$ by a lattice \mathcal{V} of “values”. In doing that, we need to reconsider the nature of A and B as well: for monotone relations, A and B were posets or preorders, for a general \mathcal{V} the objects A and B are best thought of as metric spaces with the distance measured in \mathcal{V} . This approach to metric spaces via enriched category theory was pioneered by Lawvere [35].

The basic definitions, which are instances of broader notions of enriched category theory [30], are as follows:

1. \mathcal{V} is a *commutative quantale*. This means that \mathcal{V} is a complete lattice (the lattice operations are denoted by \wedge and \vee) equipped with a commutative monoid structure (the monoid multiplication is denoted by \otimes , its unit by e) such that the multiplication is monotone in both variables and such that the equality

$$a \otimes \bigvee_{i \in I} x_i = \bigvee_{i \in I} a \otimes x_i$$

holds for any I -tuple (x_i) of elements of \mathcal{V} . The above distributive law is equivalent to the existence of “implication”, namely, $a \multimap b$ is an element of \mathcal{V} that satisfies the following property $x \leq a \multimap b$ iff $a \otimes x \leq b$, for any x .

Examples: the lattice $\mathbb{2}$ is a commutative quantale, when one defines \otimes to be \wedge and $e = 1$. In fact, every complete Heyting algebra (hence, in particular, every complete Boolean algebra) is a commutative quantale.

Another example is the extended nonnegative reals $[0; +\infty]$ with the reversed order (so that $+\infty$ is the least element), the lattice operations being given by maximum and minimum, and the tensor product $x \otimes y = x + y$. Thus, \mathcal{V} -categories are the generalised metric spaces [35].

2. Given a commutative quantale \mathcal{V} , a (small) \mathcal{V} -category A is given by a set A_0 of objects, together with the “distance” $A(a', a)$ in \mathcal{V} for any pair a', a of objects. The distance has to satisfy the following two axioms

$$e \leq A(a, a), \quad A(a'', a') \otimes A(a', a) \leq A(a'', a)$$

Examples: every poset A becomes a $\mathbb{2}$ -category if we set $A(a', a) = 1$ iff $a' \leq a$. In fact, it is easy to see that $\mathbb{2}$ -categories are precisely the preorders, since the above two axioms on distance spell out as reflexivity and transitivity.

A \mathcal{V} -category A for the quantale $[0; +\infty]$ of extended nonnegative reals is a metric space in a rather broad sense. The axioms for distance then say $A(a, a) = 0$ and $A(a'', a) \leq A(a', a) + A(a', a)$.

The quantale \mathcal{V} itself becomes a \mathcal{V} -category if we set $\mathcal{V}(x', x) = x' \multimap x$. The quantale axioms ensure that inequalities

$$e \leq x \multimap x, \quad (x'' \multimap x') \otimes (x' \multimap x) \leq x'' \multimap x$$

hold.

3. As the above terminology suggests, there should be the notion of a \mathcal{V} -functor that generalises the notion of a monotone map or a nonexpanding map. Indeed, given two \mathcal{V} -categories A, B , a \mathcal{V} -functor $f : A \longrightarrow B$ is given by the object assignment $f : A_0 \longrightarrow B_0$ subject to the axiom

$$A(a', a) \leq B(fa', fa)$$

It is then easy that \mathcal{V} -functors encode precisely the monotone mappings in case $\mathcal{V} = \mathbb{2}$ and nonexpanding mappings in case $\mathcal{V} = [0, +\infty]$.

4. Since \mathcal{V} -functors clearly compose, we have defined a category $\mathcal{V}\text{-cat}$. Moreover, $\mathcal{V}\text{-cat}$ is enriched in posets, since \mathcal{V} -functors can be compared. In ordinary category theory, a “comparison” of two functors is done by introducing natural transformations. In the case we study here, the situation is essentially the same, although “naturality” boils down to a simple condition, since the structure of \mathcal{V} is rather simple. Explicitly, given \mathcal{V} -functors $f : A \rightarrow B$, $g : A \rightarrow B$, there is at most one “natural transformation” from f to g and will be denoted as an inequality:

$$f \leq g \quad \text{iff} \quad fa \leq ga \text{ for every } a \text{ in } A$$

From now on we fix a commutative quantale \mathcal{V} . The notions of relations, exact squares, etc are straightforward generalisations of the notions for monotone relations:

1. A \mathcal{V} -valued relation $R : A \multimap B$ is a \mathcal{V} -functor $R : A^{\text{op}} \otimes B \rightarrow \mathcal{V}$. Here, A^{op} has the same objects as A , only the distance is “reversed”: $A^{\text{op}}(a', a) = A(a, a')$. And $X \otimes Y$ denotes, for \mathcal{V} -categories X and Y , the “product” having pairs (x, y) as objects with the distances $(X \otimes Y)((x', y'), (x, y)) = X(x', y') \otimes Y(y', y)$.

In what follows we will write $R(a, b)$ for the value of the functor R at (a, b) . The fact that R is a \mathcal{V} -functor can be expressed in an elementary fashion by two inequalities

$$A(a', a) \otimes R(a, b) \leq R(a', b), \quad R(a, b) \otimes B(b, b') \leq R(a, b')$$

that clearly generalise the weakening-closedness of monotone relations.

Examples: every monotone relation is a \mathcal{V} -relation. Every \mathcal{V} -category A gives rise to the *identity* relation $\text{id}_A : A^{\text{op}} \otimes A \rightarrow \mathcal{V}$ with $\text{id}_A(a', a) = A(a', a)$.

2. Given \mathcal{V} -valued relations $R : A \multimap B$ and $S : B \multimap C$, their *composite* is the relation $S \cdot R : A \multimap C$ with

$$(S \cdot R)(a, c) = \bigvee_b S(b, c) \otimes R(a, b)$$

It is easy to see that the above composition is associative and has the identity relations $\text{id}_A : A \multimap A$ as identities.

Hence we obtain a *category* $\mathcal{V}\text{-Rel}$ of \mathcal{V} -relations. Analogously to the case of monotone relations, the category $\mathcal{V}\text{-Rel}$ can be seen as enriched in posets, if we set

$$R \leq S \quad \text{iff} \quad R(a, b) \leq S(a, b), \text{ for every } a, b$$

for relations $R : A \multimap B$, $S : A \multimap B$.

3. For every \mathcal{V} -functor $f : A \rightarrow B$ there are again two ways of making a \mathcal{V} -relation out of it:

$$f_*(a, b) = B(fa, b), \quad f^*(b, a) = B(b, fa)$$

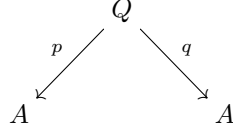
And, again, the above two “graph constructions” constitute two locally monotone functors

$$(-)_* : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-Rel}^{\text{co}}, \quad (-)^* : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-Rel}^{\text{op}}$$

So far, the many-valued notions were rather straightforward generalisations of the “monotone” notions. Alas, the next step, namely, the tabulation of relations fails.

Example 3.9. Suppose \mathcal{V} is the three-element lattice $\perp < a < \top$ with meet as the tensor product. Let A be the \mathcal{V} category on one object x with $A(x, x) = \top$ and let $R : A \multimap A$ be the relation $R(x, x) = a$. We claim that R cannot be tabulated.

Supposing the contrary, let a span

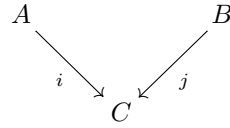


tabulate R . Then the composite $q_* \cdot p^*$ has the value

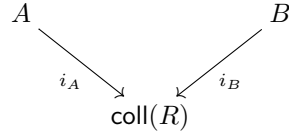
$$(q_* \cdot p^*)(x, x) = \bigvee_w A(x, pw) \wedge A(qw, x)$$

However, if Q is empty, the above join evaluates to \perp and if Q is nonempty, it evaluates to \top . Hence $(q_* \cdot p^*) \neq R$, a contradiction.

Luckily, one can prove that *cotabulations* of \mathcal{V} -relations work nicely. Thus, we say that a \mathcal{V} -relation $R : A \multimap B$ is cotabulated by a cospan



of \mathcal{V} -functors, if the equality $R = j_* \cdot i_*$ holds. There is a prominent cospan that cotabulates relation R ; its *collage*:



The \mathcal{V} -category $\text{coll}(R)$ has the disjoint union $A_0 + B_0$ as objects and the “distances” are defined as follows

$$\text{coll}(R)(a', a) = A(a', a), \quad \text{coll}(R)(b', b) = A(b', b), \quad \text{coll}(R)(a, b) = R(a, b), \quad \text{coll}(R)(b, a) = \perp$$

for all a', a in A_0 and all b', b in B_0 .

The \mathcal{V} -functor $i_A : A \rightarrow \text{coll}(A)$ sends a to a and similarly for i_B . Notice that both i_A and i_B have a special property: they are *fully faithful*. A general \mathcal{V} -functor $j : X \rightarrow Y$ is fully faithful if we have $X(x', x) = Y(jx', jx)$ for all x', x .

In fact, \mathcal{V} -functors that are fully faithful play the same role as the epimorphisms played in the set case. In fact, all of the properties of epimorphisms one needed for relation liftings, now hold for fully faithful \mathcal{V} -functors. The properties are however dualised since we work with cospans and not spans. Thus, for example:

1. A \mathcal{V} -functor j is fully faithful iff the equality $j_* \cdot j_* = \text{Id}$ holds.
2. Fully faithful functors are the “mono part” of a factorisation system on $\mathcal{V}\text{-cat}$. The corresponding “epi part” of the factorisation system contains all \mathcal{V} -functors that are cotuplings of collages of a \mathcal{V} -relation.
3. Two cospans represent the same relation if they are connected by a fully faithful \mathcal{V} -functor.

It is now straightforward to define a relation lifting of a locally monotone functor $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$ by setting

$$(\overline{T})(R) = j_* \cdot i_* \text{ where } (C, i, j) \text{ cotabulates } R$$

This definition is independent of the choice of the cospan and allows one to characterise locally monotone functors $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$ that admit a locally monotone functorial lifting $\overline{T} : \mathcal{V}\text{-Rel} \rightarrow \mathcal{V}\text{-Rel}$ as precisely

those T that preserve exactness of lax squares in $\mathcal{V}\text{-cat}$ (this is called the Beck-Chevalley condition on T), see [15, Corollary 5.11]. Note that again exact squares are the appropriate generalisation of a weak pullback.

Here, a square

$$\begin{array}{ccc}
 & W & \\
 p \swarrow & & \searrow q \\
 A & \leq & B \\
 f \searrow & & \swarrow g \\
 & C &
 \end{array}$$

of \mathcal{V} -functors is called *exact* if the equation

$$C(fa, gb) = \bigvee_w A(a, pw) \otimes B(qw, b)$$

holds for all a and b .

The universal property of $\mathcal{V}\text{-Rel}$ over $\mathcal{V}\text{-cat}$ can be formulated in an analogous way to the monotone case (notice that epi are traded for fully faithful and the condition is dualised):

Theorem 3.10 ([15]). *Suppose \mathcal{V} is a commutative quantale. The locally monotone functor $(-)_* : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-Rel}^{\text{co}}$ has the following three properties:*

1. $(-)_*$ preserves maps, that is, every f_* has a right-adjoint (denoted f^*).
2. $q_* \cdot p^* = g^* \cdot f_*$ for all exact squares

$$\begin{array}{ccc}
 & W & \\
 p \swarrow & & \searrow q \\
 A & \leq & B \\
 f \searrow & & \swarrow g \\
 & C &
 \end{array}$$

3. $j^* \cdot j_* = \text{Id}$ for every fully faithful j .

Moreover, the functor $(-)_*$ is universal w.r.t. these three properties in the following sense: if \mathcal{K} is any Pos-category to give a locally monotone functor $H : \mathcal{V}\text{-Rel} \rightarrow \mathcal{K}$ is the same as to give a locally monotone functor $F : \mathcal{V}\text{-cat} \rightarrow \mathcal{K}^{\text{co}}$ with the following three properties:

1. Every Ff has a right adjoint, denoted by $(Ff)^r$.
2. $Fq \cdot (Fp)^r = (Fg)^r \cdot Ff$ for all exact squares as above.
3. $(Fj)^r \cdot Fj = \text{Id}$ for all fully faithful j .

3.E. Relators

The notion of relator appears in different places with different meanings, but generally refers to a functor $\text{Rel} \rightarrow \text{Rel}$ with certain properties. Due to the variety of different definitions, we use relator as a generic term made precise in different ways. We give a short overview of some of the variations encountered in the literature. Roughly speaking, whereas relation liftings preserve identities, composition and order, the

various notions of relators weaken these requirements. For example, as we will see, relation liftings of **Set**-functors capture bisimulation whereas relators allows us to capture simulation (and thus are closely related to relation liftings of **Pos**-functors).

According to Abramsky and Jensen [1] a relator $\text{Rel} \rightarrow \text{Rel}$ preserves identities and maps but not necessarily composition. The emphasis of [1] is on relators as types, on terms as transformations of relators and on polymorphic invariance. The work of Backhouse and collaborators, see eg [8], also emphasises type constructors on relators and uses initial fixpoints of relators for program development in the Bird-Meertens formalism. A relator H in the sense of [8] preserves composition, inclusion and converse of relations and satisfies $H(\text{Id}) \subseteq \text{Id}$.

Most relevant from our perspective are the relators of Carboni, Kelly, and Wood [17] and Thijs [52]. To start with, note that Theorem 3.1 has as a corollary the following.

Theorem 3.11 ([17]). *There is a bijection between locally monotone functors $\text{Rel} \rightarrow \text{Rel}$ and weak pullback preserving functors $\text{Set} \rightarrow \text{Set}$.*

As it is clear from the proof of Theorem 3.1, the correspondence is as follows. Since a locally monotone functor $\text{Rel} \rightarrow \text{Rel}$ preserves adjoints it does preserve maps and restricts to **Set**. Conversely, a weak pullback preserving functor $\text{Set} \rightarrow \text{Set}$ extends via relation lifting to a locally monotone functor $\text{Rel} \rightarrow \text{Rel}$.

In the above correspondence, one can weaken the requirements and Theorem 3.2 and Remark 3.3 have the following corollary. (Here and in the following we assume that operations $H : \text{Rel} \rightarrow \text{Rel}$ are graph homomorphisms, that is, if $R : X \multimap Y$ then $HR : HX \multimap HY$.)

Theorem 3.12 ([17]). *There is a bijection between operations $H : \text{Rel} \rightarrow \text{Rel}$ satisfying*

$$\begin{aligned} (a) \quad & \text{Id}_{HA} = H(\text{Id}_A) \\ (b) \quad & g \cdot R \subseteq S \cdot f \Rightarrow Hg \cdot HR \subseteq HS \cdot Hf \\ (c_{\geq}) \quad & H(R \cdot S) \subseteq HR \cdot HS \end{aligned}$$

and functors $T : \text{Set} \rightarrow \text{Set}$.

In the two theorems above, the relators $H : \text{Rel} \rightarrow \text{Rel}$ are *tabulation-defined*, that is, they satisfy $H(cR_* \cdot dR^*) = H(cR_*) \cdot H(dR^*)$ and are determined by their action on maps, so that H and T determine each other via $Tf = Hf_*$.

But, as we will see now, there are good reasons to be interested in relators that are not tabulation-defined. Recall Example 2.8 of the powerset functor. The ‘back and forth’ conditions $(\forall x \in a \exists y \in b \dots)$ & $(\forall y \in b \exists x \in a \dots)$ describing $\overline{\mathcal{P}}$ correspond, as we will see in the next section, to bisimulation. But any of the two conjuncts on its own gives a good notion of simulation. The idea of how to weaken tabulation-defined relators in order to account for simulation is due to [52] and illustrated in the following

Example 3.13. Let $R : X \multimap Y$ and $a \in \mathcal{P}X$ and $b \in \mathcal{P}Y$. We have

$$\begin{aligned} \forall x \in a. \exists y \in b. xRy & \iff (a, b) \in \subseteq \cdot \overline{\mathcal{P}}(R) \cdot \subseteq \\ \forall y \in b. \exists x \in a. xRy & \iff (a, b) \in \supseteq \cdot \overline{\mathcal{P}}(R) \cdot \supseteq \end{aligned}$$

This suggests that whereas bisimulation arises from tabulation-defined relators, simulation arises from pre- and post-composing tabulation-defined relators with preorders. If $T : \text{Set} \rightarrow \text{Set}$ factors through a functor $G : \text{Set} \rightarrow \text{Preord}$ as $T = VG$ with $V : \text{Preord} \rightarrow \text{Set}$ then G is called an extension of T in [52] and an **order on T** in [25]. (The definition of extension in [52, Def.2.2.1] has an extra condition which reflects the fact that all functors in [52] are standard, see Remark 2.13.)

The next proposition summarises the properties of the relator H induced by an order G on T .

Proposition 3.14 ([52]). *Let $G : \text{Set} \rightarrow \text{Preord}$ be an order on $T : \text{Set} \rightarrow \text{Set}$. Then mapping $R : X \multimap Y$ to*

$$HR = GX \cdot \bar{T}(R) \cdot GY$$

satisfies

$$\begin{aligned} (a_{\leq}) \quad & \text{Id}_{TA} \subseteq H(\text{Id}_A) \\ (c_{\geq}) \quad & H(R \cdot S) \subseteq HR \cdot HS \\ (m) \quad & R \subseteq S \Rightarrow HR \subseteq HS \\ (e) \quad & Tf_* \cdot H(R) \cdot Tg^* \subseteq H(f_* \cdot R \cdot g^*) \end{aligned}$$

Comparing the relators $H : \text{Rel} \rightarrow \text{Rel}$ satisfying $(a_{\leq}), (c_{\geq}), (m), (e)$ with the relators of Theorem 3.12, we find that the only difference is that preservation of identities is weakened. To make the comparison easier, we state the following lemma, see [17, Section 2].

Lemma 3.15. *Let $T : \text{Set} \rightarrow \text{Set}$ be a functor and $H : \text{Rel} \rightarrow \text{Rel}$ an operation. Then*

$$f_* \cdot R \subseteq S \cdot g_* \Rightarrow Tf_* \cdot HR \subseteq HS \cdot Tg_*$$

is equivalent to H being locally monotone plus any of the following

$$\begin{aligned} Tf_* \cdot H(R) \cdot Tg^* &\subseteq H(f_* \cdot R \cdot g^*) \\ H(f_* \cdot S \cdot g_*) &\subseteq Tf^* \cdot HS \cdot Tg_* \end{aligned}$$

The last two conditions are often written as

$$\begin{aligned} (Tg \times Tf)[H(R)] &\subseteq H((g \times f)[R]) \\ H((g \times f)^{-1}S) &\subseteq (Tg \times Tf)^{-1}HS \end{aligned}$$

If we strengthen condition (c_{\geq}) to equality, we obtain a theorem similar to Theorems 3.11 and 3.12. It is essentially [52, Thm 2.2.3] with the condition on the order taken from [36] and [25].

Theorem 3.16. *Given $T : \text{Set} \rightarrow \text{Set}$ there is a bijection between operations $H : \text{Rel} \rightarrow \text{Rel}$ satisfying*

$$\begin{aligned} (a_{\leq}) \quad & \text{Id}_{TA} \subseteq H(\text{Id}_A) \\ (c) \quad & H(R \cdot S) = HR \cdot HS \\ (m) \quad & R \subseteq S \Rightarrow HR \subseteq HS \\ (e) \quad & Tf_* \cdot H(R) \cdot Tg^* \subseteq H(f_* \cdot R \cdot g^*) \end{aligned}$$

and orders G on T mapping weak pullbacks to exact squares.

The theorem characterises T -relators in terms of the induced order G on T . Alternatively, one can extend G to a functor $T' : \text{Preord} \rightarrow \text{Preord}$ and then consider the relation lifting $H' : \text{Rel}(\text{Preord}) \rightarrow \text{Rel}(\text{Preord})$ of Section 3.C. A categorical characterisation of which functors $\text{Preord} \rightarrow \text{Preord}$ arise in this way is given by [9, Thm 4.13] (replacing Pos by Preord). As observed in [52, 36, 39] many important properties of relators still work if one insists only on lax preservation of composition:

Definition 3.17. A T -relator H is an operation that maps relations $R : A \multimap B$ to $HR : TA \multimap TB$ subject to

$$\begin{aligned} (a_{\leq}) \quad & \text{Id}_{TA} \subseteq H(\text{Id}_A) \\ (c_{\leq}) \quad & HR \cdot HS \subseteq H(R \cdot S) \\ (m) \quad & R \subseteq S \Rightarrow HR \subseteq HS \\ (e) \quad & Tf_* \cdot H(R) \cdot Tg^* \subseteq H(f_* \cdot R \cdot g^*) \end{aligned}$$

It is easy to see (a_{\leq}) and (c_{\leq}) guarantee that $H(\text{Id}_A)$ is a preorder and, due to (e) , that $Tf : (X, H(\text{Id}_X)) \rightarrow (Y, H(\text{Id}_Y))$ is monotone for all $f : X \rightarrow Y$. Thus, similarly to the situation described in the previous theorem, a T -relator allows us to extend T to a functor $T' : \text{Preord} \rightarrow \text{Preord}$. We refer to [36] for more information and for an example of a relator that does not preserve composition. [39] shows that a finitary functor T has a separating set of monotone predicate liftings iff T has a ‘lax-relation lifting preserving diagonals’, ie a T -relator in the sense of Definition 3.17 satisfying (a) . This usage of ‘lax’ is different from [25], see Lemma 5.3 op.cit.

4. Applications

We present two applications. In the first, one defines (bi)simulations via relators. By varying the base category from sets to preorders or even to (generalised) metric spaces one goes from bisimulation via simulation to metric simulation. For this, one does not need that relators preserves composition. In the second application relation lifting is used to define the semantics of Moss’s ‘cover modality’ ∇_T capturing the logical content of the type-functor T . Here, preservation of composition plays a role in order to obtain invariance of the logic under (bi)similarity.

4.A. Simulation and bisimulation

According to the Park-Milner definition of bisimulation, a bisimulation is a post-fixed point and bisimilarity is the greatest fixed point, see Remark 4.2 below and [46] for the history and further references. Based on the notion of relation lifting, the Rutten [44] generalises this to coalgebras. In the discrete situation of Set it gives bisimulation, in the ordered setting of Preord or Pos it gives simulation and for \mathcal{V} -cat one obtains a metric version studied in detail by Worrell [56].

Bisimulations arise from relation lifting for endo-functors on Set .

Definition 4.1. Let $T : \text{Set} \rightarrow \text{Set}$ be a functor and $\xi : X \rightarrow TX$ and $\xi' : X' \rightarrow TX'$. Then R is a T -bisimulation if

$$\begin{array}{ccc}
 X & \xrightarrow{\xi} & TX \\
 R \downarrow & \subseteq & \downarrow \overline{TR} \\
 X' & \xrightarrow{\xi'} & TX'
 \end{array} \tag{20}$$

Remark 4.2. It is useful to note the following equivalent ways of rendering (20)

1. $\xi'_* \cdot R \subseteq \overline{TR} \cdot \xi_*$
2. $R \subseteq \xi'^* \cdot \overline{TR} \cdot \xi_*$
3. $xRx' \Rightarrow \xi(x) (\overline{TR}) \xi'(x')$
4. $R \subseteq ((\xi \times \xi')^{-1} \circ \overline{T})(R)$

the last of which emphasises that R is a post-fixed point of the monotone operator $(\xi \times \xi')^{-1} \circ \overline{T}$.

Spelling out the leading examples, we find that this definition gives the expected results. For example, use item (3) of the remark above together with Example 2.8 to obtain:

Example 4.3. Given $x \in X \xrightarrow{\xi} \mathcal{P}X$ and $x' \in X' \xrightarrow{\xi'} \mathcal{P}X'$, we have that R is a bisimulation iff

$$xRx' \Rightarrow (\forall y \in \xi(x). \exists y' \in \xi'(x'). x'Ry') \ \& \\ (\forall y' \in \xi'(x'). \exists y \in \xi(x). x'Ry')$$

Remark 4.4. From the point of our exposition, the “ \subseteq ” in (20) is a 2-cell in a 2-category where relations are arrows. From another point of view, one can also interpret the “ \subseteq ” as a lifted coalgebraic structure where relations are objects in a fibered category and inclusions are arrows. We don’t pursue this very interesting alternative but refer to Hermida and Jacobs’s original [23] and to [32, 16] for examples of further work in this direction.

As remarked above it is tempting to view $R \subseteq \overline{TR}$ as a coalgebra. A related, but again different idea is expressed by the following definition, saying that a relation is a bisimulation if it can be equipped with a coalgebra structure making the projections into homomorphisms.

Definition 4.5 (Aczel-Mendler [4]). $R \subseteq X \times X'$ is a *AM-bisimulation* between coalgebras (X, ξ) and (X', ξ') if there is a coalgebra structure ρ on R such that the projections $X \leftarrow R \rightarrow X'$ become coalgebra morphisms

$$\begin{array}{ccccc} X & \longleftarrow & R & \longrightarrow & X' \\ \xi \downarrow & & \downarrow \rho & & \downarrow \xi' \\ TX & \longleftarrow & TR & \longrightarrow & TX' \end{array}$$

A comparison of this definition with (20) shows that the two definitions differ only by using, respectively, TR or \overline{TR} . The following proposition now follows directly from the fact that the tabulation of $\overline{T}(R)$ is constructed by epi-mono factoring the TR -span, plus the fact that epis split in \mathbf{Set} . (In general the two definitions are not equivalent, see [50] for a detailed comparison of coalgebraic notions of bisimulation.)

Proposition 4.6. R is a bisimulation iff R is an AM-bisimulation.

We have seen two equivalent definitions of bisimulation. Another way to obtain the notion of bisimilarity is via the final coalgebra. The final coalgebra has not played an important role in this paper so far. But it is a powerful device to provide semantic domains for processes as known, for example, from process algebra. In domain theory the initial algebra has been used more than the final coalgebra, but then in many of the categories typical in domain theory the famous limit-colimit coincidence [2] means that initial algebras and final coalgebras agree. It was Aczel’s insight [3] that in categories where this does not happen, as eg in the category \mathbf{Set} , it is the final coalgebra that provides the right semantic domain for recursively defined data or processes.

Remark 4.7 (Existence of a final coalgebra). It is well known from the proof of the adjoint functor theorem [37] that the final coalgebra can be computed as a coproduct over all coalgebras quotiented by bisimilarity. Another description of essentially the same construction is obtained by taking the colimit of the forgetful functor $U : \mathbf{Coalg}(T) \rightarrow \mathbf{Set}$. In both cases the colimit is taken over a large diagram and therefore does not need to exist in $\mathbf{Coalg}(T)$. There are two common solutions to this problem.

The first is to require that T is accessible which means that T is determined by its action on sets smaller than some regular cardinal. For example, the final coalgebra for the powerset functor \mathcal{P} does not exist in $\mathbf{Coalg}(\mathcal{P})$ but the final \mathcal{P}_ω -coalgebra does exist in $\mathbf{Coalg}(\mathcal{P}_\omega)$. This remains true if we replace ω by an inaccessible cardinal, giving us a final coalgebra for the powerset-functor [11].

The second, and equivalent solution, is to enlarge the universe \mathbf{Set} to a bigger universe \mathbf{Set}' and extend $T : \mathbf{Set} \rightarrow \mathbf{Set}$ to a functor $T' : \mathbf{Set}' \rightarrow \mathbf{Set}'$ which does admit a final coalgebra. This universe enlargement is always possible [30].

Both solutions introduce a distinction between large and small sets and, consequently, between large and small coalgebras. The coalgebras of interest are the small ones, but the final coalgebra is large. What one

needs to show then is that the large final coalgebra classifies the bisimilarity of the small coalgebras. In essence, this means that every element of the final coalgebra lives in a small subcoalgebra. This is indeed the case and was first shown in [4]. For further developments we refer to [5, 6]. \square

To summarize the remark above, in the following we feel free to use a final coalgebra whenever convenient.

Theorem 4.8 (coinduction theorem). *Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ preserve weak pullbacks. Let $X \rightarrow TX$ be a coalgebra and $!$ the coalgebra morphism from X into the final coalgebra. Then $!(x) =!(y)$ iff there is a bisimulation R with xRy .*

For the proof we refer to Rutten [45] and only say where the weak-pullback preservation comes in. In order to exhibit the bisimulation required by the ‘if’ direction, one takes as R the kernel of $!$. Since, by virtue of being a kernel, R is a pullback, one obtains the required $\rho : R \rightarrow TR$ as the (not necessarily unique) arrow into the weak pullback TR .

Simulations arise from relation lifting of endofunctors on the category \mathbf{Preord} of preorders, or the category \mathbf{Pos} of posets, much in the same way as bisimulations arise from endofunctors on \mathbf{Set} . Our main sources are here the following. Worrell [56], following a suggestion of Rutten [44], subsumes \mathbf{Preord} under the more general \mathcal{V} -category case, Hughes and Jacobs [25] present a detailed study dedicated to \mathbf{Preord} and Levy [36] focusses on \mathbf{Pos} . Conceptually, the difference between \mathbf{Preord} and \mathbf{Pos} is that a preordered final coalgebra allows to account for similarity (via the order) and bisimilarity (via equality) simultaneously, reflecting the well-known phenomenon that mutual similarity is weaker than bisimilarity. Here, we decided to follow [36] but concentrate on preorders instead of posets.

The definition of **simulation** in the new setting is still given by (20), just that now we let $T : \mathbf{Preord} \rightarrow \mathbf{Preord}$. Note that below we consider simulations on a coalgebra, that is we let $(X, \xi) = (X', \xi')$ in (20).

For $X \in \mathbf{Preord}$ we denote by X_o the underlying set and by \leq_X the corresponding preorder.

Proposition 4.9. 1. *Given a coalgebra $X \rightarrow TX$, the preorder \leq_X is a simulation.*

2. *The union of simulations is a simulation and for every coalgebra $X \rightarrow TX$ there is a largest simulation \sqsubseteq_X .*

3. *Given a coalgebra $X \rightarrow TX$ the quotient wrt the largest simulation on X exists and is given by $(X_o, \sqsubseteq_X) \rightarrow T(X_o, \sqsubseteq_X)$.*

4. *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are coalgebra morphisms and if $R : X' \rightarrow Y'$ is a simulation, then $g^* \cdot R \cdot f_* : X \rightarrow Y$ is a simulation. Moreover, if R is the largest simulation, so is $g^* \cdot R \cdot f_*$.*

Proof. (1) follows from \bar{T} preserving identities and (2) follows from \bar{T} preserving inclusion of relations. (3) requires more work, but follows arguments well-known for coalgebras, see eg [36]. (4) follows from properties of relation lifting. \square

Due to item (3) in the proposition above, the final T -coalgebra exists, even though its carrier might possibly be a proper class, see Remark 4.7.

Theorem 4.10. (coinduction theorem) *Let $T : \mathbf{Preord} \rightarrow \mathbf{Preord}$. Let $X \rightarrow TX$ be a coalgebra and $!$ the coalgebra morphism from X into the final coalgebra $Z \rightarrow TZ$. Then $!(x) \leq_Z !(y)$ iff there is a simulation R with xRy .*

Remark 4.11. It is possible to develop the above for \mathbf{Pos} instead of \mathbf{Preord} , see [36]. Simulations in the \mathbf{Preord} -sense include bisimulations in the \mathbf{Set} -sense. Technically, in our framework, this is shown by Theorem 4.3 of [9] (which holds for \mathbf{Preord} for the same reasons as it does hold for \mathbf{Pos}): Every \mathbf{Set} functor T can be extended in a canonical way to a \mathbf{Preord} -functor T' and on discrete coalgebras T -bisimulations coincide with T' -simulations. To make the connection with the relators of Section 3.E, we recall that they give rise to functors $\mathbf{Preord} \rightarrow \mathbf{Preord}$. Accordingly we use the monotone relation lifting along $\mathbf{Preord} \rightarrow \mathbf{Rel}(\mathbf{Preord})$ to define simulation as in (20).

Metric simulations arise from the relation lifting of endofunctors on \mathcal{V} -categories presented in Section 3.D and our development, closely following Worrell [56], is now based on relation lifting via cospans.

The definition of simulation in the new setting is still given by (20), just that now we let $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$ and replace “ \subseteq ” by \leq , referring to the order on \mathcal{V} . Consequently, relying on the notation of Remark 4.2 (see, in particular, item (3)), R is a simulation iff

$$R(x, x') \leq \bar{T}R(\xi(x), \xi'(x'))$$

Proposition 4.12. *1. Given a coalgebra $X \rightarrow TX$, the $X(x, x')$ is a simulation.*

2. The join of simulations is a simulation and for every coalgebra $X \rightarrow TX$ there is a largest simulation \sqsubseteq_X .

3. If T preserves embeddings, then simulations are closed under composition.

4. If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are coalgebra morphisms and if $R : X' \multimap Y'$ is a simulation, then $g^ \cdot R \cdot f_* : X \multimap Y$ is a simulation. Moreover, if R is the largest simulation, so is $g^* \cdot R \cdot f_*$.*

The next theorem specialises in the case of $\mathcal{V} = \mathbf{2}$ to a theorem on preorders similar to Theorem 4.10 but it is formulated in such a way that it accounts for the possibility of measuring ‘distances’ expressed by the values in \mathcal{V} .

Theorem 4.13. *(coinduction theorem) Let $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$ preserve embeddings. Let $X \rightarrow TX$ be a coalgebra and $!$ the coalgebra morphism from X into the final coalgebra $Z \rightarrow TZ$. Then $Z(!x, !x') = \bigvee \{R(x, x') \mid R \text{ a simulation on } X \rightarrow TX\}$.*

4.B. Coalgebraic logic

Given a functor $T : \text{Set} \rightarrow \text{Set}$ describing the possible one-step behaviours of coalgebras $X \rightarrow TX$, can we find modal operators to specify the behaviour of coalgebras?

For overviews summarising and comparing different answers to this question, we refer to [34, 18]. Here, we want to single out one important approach, due to Moss [41], which is based on the relation lifting and makes crucial use of the relation lifting preserving composition.

Let us assume we have a set of formulas \mathcal{L} and a semantics \Vdash_ξ w.r.t. coalgebras $\xi : X \rightarrow TX$ and states x in X given as a relation

$$\Vdash_\xi \subseteq X \times \mathcal{L}$$

The idea of Moss’s logic is to only use one modal operator, which nowadays is written as ∇ and pronounced ‘nabla’, and which is formed according to

$$\frac{\gamma \in T\mathcal{L}}{\nabla\gamma \in \mathcal{L}}$$

The semantics of ∇ is obtained by lifting the satisfaction relation \Vdash_ξ as follows (writing \Vdash for \Vdash_ξ)

$$x \Vdash \nabla\gamma \Leftrightarrow \xi(x) \bar{T}(\Vdash) \gamma. \quad (21)$$

Example 4.14. Let $T = \mathcal{P}$ and $\Phi \subseteq \mathcal{L}$ a set of formulas. Then $x \Vdash \nabla\{\phi_1, \dots, \phi_n\}$ iff all successors of x satisfy some formula ϕ_i and all formulas ϕ_1, \dots, ϕ_n are satisfied by some successor of x . Note the “forall-exists-and-forall-exists” pattern induced by the relation lifting of the powerset and familiar from Examples 2.8 and 4.3.

One of the basic properties of the logic is invariance under bisimilarity if T preserves weak pullbacks.

Proposition 4.15. *If T preserves weak pullbacks and $R : X' \multimap X$ is a bisimulation between two coalgebras $X' \multimap TX'$ and $X \multimap TX$ and if $\Vdash \cdot R \subseteq \Vdash'$, then $x \Vdash \nabla \gamma \Rightarrow x' \Vdash' \nabla \gamma$.*

Proof. The data of the proposition is depicted in

$$\begin{array}{ccc}
 TX' & \xrightarrow{\bar{T}(R)} & TX & \xrightarrow{\bar{T}(\Vdash)} & T\mathcal{L} \\
 \uparrow \xi'_* & & \uparrow \xi_* & & \\
 X' & \xrightarrow{R} & X & \xrightarrow{\Vdash} & \mathcal{L}
 \end{array} \tag{22}$$

Assuming $x \Vdash \nabla \gamma$, that is $x (\bar{T}(\Vdash) \cdot \xi_*) \gamma$, we have $x' (\bar{T}(\Vdash) \cdot \xi_* \cdot R) \gamma$ and by R being a bisimulation $x' (\bar{T}(\Vdash) \cdot \bar{T}(R) \cdot \xi'_*) \gamma$, holds. Hence, by preservation of weak pullbacks, $x' (\bar{T}(\Vdash \cdot R) \cdot \xi'_*) \gamma$, which implies $x' (\bar{T}(\Vdash') \cdot \xi'_*) \gamma$, that is $x' \Vdash' \nabla \gamma$. \square

Monotone coalgebraic logic has been investigated in [14, 15]. The categories **Preord** and **Pos** are special in the sense that monotone relations can both be represented by spans and cospans. We have seen the treatment via spans in Section 3.C and, by specialising to $\mathcal{V} = \mathbb{2}$, the treatment via cospans in Section 3.D. In both cases one obtains the same semantics for ∇ and so we proceed immediately to the case of a general \mathcal{V} -cat.

Metric coalgebraic logic follows the same pattern as laid out for **Set** above, but instantiating the definitions in the setting of Section 3.D gives a much richer structure.

One question that didn't arise in the discrete setting and that needs attention is what type we want to give the satisfiability relation. If the order on X should represent information order one would like to have that interpretations of formulas are upsets ordered by *reverse inclusion*. It therefore makes sense to stipulate that satisfaction is given by a relation

$$\Vdash : X^{\text{op}} \multimap \mathcal{L}$$

Then, in order to match the types correctly in (22), we see that we need to replace T by its dual T^∂ defined by $T^\partial(X) = (T(X^{\text{op}}))^{\text{op}}$ and X and ξ by X^{op} and ξ^{op} . After this little modification the proof of Proposition 4.15 goes through again.

Example 4.16. Let $TX = \mathcal{U}X = [X, \mathcal{V}]^{\text{op}}$, that is, TX is the set of ‘upsets’ of X ordered by reverse inclusions. Note that $T^\partial X = \mathcal{D}X = [X^{\text{op}}, \mathcal{V}]$.

$$\begin{array}{ccc}
 X & & \mathcal{L} \\
 \searrow d & & \swarrow c \\
 & \Vdash &
 \end{array}$$

$$\begin{aligned}
 (T^\partial c)^* \cdot (T^\partial d)_* (\xi(x), \alpha) &= [\Vdash^{\text{op}}, \mathcal{V}](\mathcal{D}(d)(\xi(x)), \mathcal{D}(c)(\alpha)) \\
 &= [\Vdash^{\text{op}}, \mathcal{V}](\xi(x), [d^{\text{op}}, \mathcal{V}] \cdot \mathcal{D}(c)(\alpha)) \\
 &= \bigwedge_{y \in \xi(x)} (\xi(x)(y) \multimap \bigvee_{\phi \in \mathcal{L}} \alpha(\phi) \otimes \Vdash(y, \phi)) \\
 &= \sup_{y \in \xi(x)} (\inf_{\phi \in \mathcal{L}} (\alpha(\phi) + y \Vdash \phi) - \xi(x)(y))
 \end{aligned}$$

where \multimap is the implication (closed structure) corresponding to \otimes . As a side remark, we would like to point out that this calculation, which is specific to the particular functor T , is algebraic in the sense that it can be performed by manipulating a precise set of rules. Coming back to evaluating the result of the calculation, we find a semantics for ∇ that resembles the \square operator of modal logic, but is much richer. To see that \square is indeed a special case, let $\alpha = \{\phi\}$ and let ϕ be crisp (taking values in $\{0, \infty\}$) and X be discrete. Then

$$(T^\partial c)^* \cdot (T^\partial d)_* (\xi(x), \alpha) = \forall y \in \xi(x). y \Vdash \phi$$

that is $\nabla = \square$.

Dually, starting with $T = \mathcal{D}$, one obtains a generalisation of the diamond operator from modal logic. It is quite pleasant to see that this relationship between \mathcal{U}/\Box and \mathcal{D}/\Diamond well-known in domain theory [54] arises here from the general machinery in the case of $\mathcal{V} = \mathbb{2}$ and is appropriately generalised for more general \mathcal{V} .

5. Conclusions

To summarize, we have seen how to lift relations to a large number of data types. In case these data types are polynomial (that is, built from (co)products of sets) this is straightforward, but for other datatypes such as powerdomains or probability distributions, interesting notions of bisimulation arise. Furthermore, ordered and metric versions of these notions are obtained by moving from sets to orders or to \mathcal{V} -categories.

From placing this survey in the context of RAMICS, a number of questions arise.

For example, Chapter 7 of Schmidt [47] shares with the category theoretic approach the emphasis on type constructors (such as product, coproduct, powerset, etc), as well as the insistence on modularity for building complex data types and for proving properties about them. But the methods are different, relation algebraic on the one hand, on the other functoriality, naturality and universal properties. While [47] points out that categorical constructions may not be constructive, in practice, as in this paper, they often are. A deeper comparison of the different schools of thought and their respective advantages could be a worthwhile endeavour.

Apart from developing the theory uniformly in the type functor, there might also something to be gained from looking at the work on many-valued relation algebra, see eg [28, 42, 55], from the point of view of enriched category theory where not only the relations are fuzzy but also the sets may carry an order or metric that has to be respected by the relations.

For example, what is the relation algebra of monotone relations? It will be different from allegories [19], as *Preord* and *Pos* are not regular categories. A more concrete observation here is that complement and converse have different types in the ordered setting. Indeed, given $R : A \multimap B$, the converse R^\top is of type $B^{\text{op}} \multimap A^{\text{op}}$ and the complement \bar{R} is of type $A^{\text{op}} \multimap B^{\text{op}}$, so that expressions such as $R; R^\top$ and many relation-algebraic laws (including the modularity law [19] of allegories) are not type correct anymore. Nevertheless, the Schröder equivalences⁴, see eg [48, 2.3.4], still work, showing that complement and converse could still play a role.

Finally, in a wider sense, what we presented here can be seen as part of the larger topic of Reynolds' parametricity [24]. The reason that the work on parametricity has a quite different technical flavour than the work reviewed in this paper is the following. Most work in coalgebra has focussed on *covariant datatypes* (including all datatypes seen in this paper), because they can be modelled in standard categories such as sets, orders, vector spaces, etc. But for work in parametricity the mixed-variant datatype of function spaces $X \mapsto X^X$ is of central importance, requiring the more sophisticated categories of domain theory. This suggests that there is more to gain from weaving these two strands of research together.

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⁴For relations R, S, T of appropriate type, one has $R; S \subseteq T \Leftrightarrow R \subseteq \overline{\bar{T}}; S^\top \Leftrightarrow S \subseteq \overline{\bar{R}^\top}; \bar{T}$.

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