A Short Introduction to Quantale Enriched Categories

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Abstract

A short course on quantale enriched categories written by Alexander Kurz based on conversations with Adriana Balan, Peter Jipsen and Samuel Kagan. The theory of categories enriched over a quantale broken down into a series of exercises for advanced undergraduate students.

Contents

1 Introduction

We are interested in many-valued logics with more truth values than just *true* and *false*. This is an inquiry into the question which mathematical structures support many-valued reasoning, much in the same way as Boolean algebra supports 2-valued reasoning. Our approach is algebraic and axiomatic, that is, we start with a set of truth values, which, following tradition, we denote by Ω and a list of properties we think we definitely need. Then we explore the structures we obtain via examples and by proving properties we find of interest.

To start with, as for Boolean algebras, we will require that the set Ω of all truth values has *and* and *or*, in other words, we will require that Ω is a lattice. We will write $\bot, \top$ for the bottom and top element (if they exists) and $\vee$ and $\wedge$ for join and meet, respectively. [1]

Exercise 1.1. Learn the basic facts about lattices, eg from Wikipedia. Make a list of examples of lattices.

Remark 1.2. We will denote the empty join, or bottom element, by $\bot$ (as opposed to 0) and the empty meet, or top element, by $\top$ (as opposed to 1).

In many-valued logics, we often have more operations than *and*, *or*, *not*. For example, we may want to add or multiply truth values. We capture this abstractly as follows. In addition to being a lattice, we will require that Ω is equipped with another binary operation called multiplication or fusion. We will write it as $\otimes$ and often pronounce it as "tensor". One of the classic examples is the following.

Example 1.3. Let Ω be the set of non-negative real numbers with infinity, ordered by $\geq$, that is with 0 at the top and ∞ at the bottom. $\otimes$ is addition.

Remark 1.4. In the example above we can think of the real numbers both as distances and as truth values. Distance zero corresponds to true. The reason is that the predicate $x = y$ is true if and only if the distance of x and y is 0.

Before we start looking at further examples, we need to finish the list of requirements we will put on Ω. The remaining requirements are that Ω has all joins, not only finite ones, and that $\otimes$ preserves joins coordinatewise. Taking everything together, these structures are known as quantales.

Definition 1.5. A (unital) quantale Ω is a complete lattice with an associative operation $\otimes$ that has a neutral element e and preserves all joins coordinatewise. A quantale is called commutative if $\otimes$ is commutative.

Remark 1.6. The formulation "preserves all joins coordinatewise" includes the preservation of empty joins, that is, $\bot \otimes v = \bot$ and $v \otimes \bot = \bot$.

Example 1.7. Show that a Boolean algebra becomes a commutative quantale if we choose $\otimes = \wedge$.

Example 1.8. Show that the set of languages over an alphabet becomes a quantale when we define $L \otimes L' = \{ww' \mid w \in L, w' \in L'\}$.

Example 1.9. Show that the set of binary relations on a set X forms a unital quantale when we take composition of relations as $\otimes$.

Exercise 1.10. In which of the examples above is the neutral element different from the top element?

The following definition will play a central role.

Definition 1.11. In every quantale we can define the operations $w/v = \bigvee\{x \mid x \otimes v \leq w\}$ and $v\backslash w = \bigvee\{x \mid v \otimes x \leq w\}$.

[1] "join" is defined as least upper bound and "meet" is defined as greatest lower bound.

Remark 1.12. If $\otimes$ is commutative, then $w/v = v\backslash w$ and we write $[v, w]$ instead.

Example 1.13. To justify the notation w/v, let Ω be the positive reals in their usual order with $\otimes$ being multiplication.[2] Show that $[v, w]$ is the usual division w/v of real numbers.

Exercise 1.14. In the Examples 1.3, 1.7, 1.8, 1.9, what are w/v and $v\backslash w$?

The basic property relating $\otimes$ and residual is:

Proposition 1.15. *In a commutative quantale we have that* $x \otimes v \leq w \iff x \leq [v, w]$.

Exercise 1.16. Prove the proposition above. Generalise it to the non-commutative setting.

The next proposition shows that many properties that are familiar from the examples hold for quantales in general.

Proposition 1.17. *Let* $(\Omega, e, \otimes)$ *be a commutative quantale.*

1. $[a, \top] = \top$
2. $[a, b \wedge c] = [a, b] \wedge [a, c]$
3. $[\bot, a] = \top$
4. $[b \vee c, a] = [b, a] \wedge [c, a]$
5. $[e, a] = a$
6. $e \leq [x, x]$

Exercise 1.18. Which well-known rules of logic correspond to each of the items above?

Exercise 1.19. Prove the proposition above.

It is important to know that all operations behave well wrt the order:

Exercise 1.20. Show that

1. $x \leq y \Rightarrow x \otimes a \leq y \otimes a$
2. $x \leq y \Rightarrow a \otimes x \leq a \otimes y$
3. $x \leq y \Rightarrow [a, x] \leq [a, y]$
4. $x \leq y \Rightarrow [y, a] \leq [x, a]$

In other words, $\otimes$ is monotone (ie order-preserving) in both arguments, while the residuals are monotone in the 'enumerator' and anti-tone (ie order-reversing) in the 'denominator'.

The importance of the operations w/v, $v\backslash w$, and $[v, w]$ is that they are a kind of inverse of $\otimes$. Remember how you learned to solve equations such as $x \cdot 2 = 6$ by dividing by 2 on both sides to get $x = 6/2$? The purpose of the next proposition is to show that we have a similar law in quantales.

Exercise 1.21. Conversely, show that if there is an operation $[v, w]$ satisfying $x \otimes v \leq w \iff x \leq [v, w]$ then $[v, w] = \bigvee\{x \mid x \otimes v \leq w\}$.

Proposition 1.22. *Let* $(\Omega, e, \otimes)$ *be a commutative quantale.*

1. $a \otimes [a, b] \leq b$
2. $a \leq [b, a \otimes b]$

[2] This is not a quantale, since it does not have all joins, but the joins needed for the definition of $[v, w]$ do exist.

3. $a \leq [[a, b], b]$

Exercise 1.23. Prove the proposition above using $x \otimes v \leq w \;\Leftrightarrow\; x \leq [v, w]$. Conversely, show that 1 and 2 imply $x \otimes v \leq w \;\Leftrightarrow\; x \leq [v, w]$.

Finally, I recommend that you summarize what you have learned:

Exercise 1.24.

1. Write out explicitly the equations defining a quantale.
2. Read about quantales here and here. Make a note of anything that was not already covered in this section.
3. Make a list of all examples of quantales you know.

2 Categories Enriched over Quantales

Categories enriched over quantales are systems of relations indexed by the elements of a quantale. We will define this notion only in the special case of interest to us here. While the below gives a correct definition of a category enriched over a quantale it is not the official one. If I want to emaphsise that I am referring to the definition below, I will speak about **sytsems of relations**. They are sometimes known in the literature as relational presheaves, but it would be too much of a distraction to explain this terminology. Later in the exercises to this section, we will recover the official definition of an enriched category in the special case of enriching over a quantale.

Definition 2.1 (system of relations over a quantale)**.** A system X of relations over a quantale $(\Omega, e, \otimes)$ is a set X together with a relation R_v for each $v \in \Omega$ such that

1. $\mathsf{Id} \subseteq R_e$
2. $R_v \,;R_w \subseteq R_{v \otimes w}$
3. $v \leq w$ implies $R_v \supseteq R_w$
4. $R_{\bigvee_{i \in I} v_i} = \bigcap_{i \in I} R_{v_i}$

where ";" denotes relational composition and $\mathsf{Id} = \{(x, x) \mid x \in X\}$ denotes the identity relation. The last item implies $R_\bot = X \times X$ if we choose $I = \emptyset$.

Example 2.2. Let $2 = \{0, 1\}$. Let Ω be the quantale $(2, 1, \wedge)$. Explain why we can say that categories enriched over 2 are exactly preorders, that is, relations $R \subseteq X \times X$ satisfying xRx and $xRy \;\&\; yRz \Rightarrow xRz$. More precisely, for a fixed set X, describe two functions, one from the class of systems of relations over X to the class of preorders over X, the other going in the opposite direction, and show that they are inverse to each other.

We will see more examples shortly in the next section. Nevertheless, you may want to pause here and try to manufacture your own examples, just to get a feel for this quite complicated definition.

Exercise 2.3 (Optional)**.** Try to find some examples (and maybe interesting non-examples) of the definition above.

Whether you have done the exercise above or not, it is time now to discuss the alternative description of systems of relations as enriched categories.

Remark 2.4. We can describe a system of relations X by a single Ω-valued relation, that is, a function $X \times X \to \Omega$, which is traditionally also denoted by X. Given a system of relations as in the definition above, we define

$$X(x, y) = \bigvee \{v \mid (x, y) \in R_v\}$$

Conversely, every $X(x,y)$ satisfying the usual definition of a category enriched over Ω (see below), can be turned into an equivalent system of relations by defining

$$R_v = \{(x,y) \mid v \le X(x,y)\}$$

Example 2.5. Continuing from Example 1.3 where $\Omega = [0,\infty]$, we have for all $\epsilon \ge_{\mathbb{R}} 0$ a relation $R_\epsilon = \{(x,y) \mid X(x,y) \le_{\mathbb{R}} \epsilon\}$ that contains all pairs of points that at a "distance" $\le_{\mathbb{R}} \epsilon$. The distance between two points $X(x,y) = \inf\{\epsilon \mid (x,y) \in R_\epsilon\}$ is the smallest ϵ such that (x,y) is contained in R_ϵ.

We now investigate more closely in which sense the two kind of structures we have seen in the remark and example above are equivalent.

Exercise 2.6. In the remark above, show that starting from a system of relations R, going to $X \times X \to \Omega$ and then back returns indeed a system of relations that is identical to the one we started with.

The next question is whether we can characterise the maps $X \times X \to \Omega$ that arise from a system of relations X. You may want to pause here and think about this for yourself.

... pause ...

The first step is to show the following.

Exercise 2.7. Let X be a system of relations over a commutative quantale Ω. Show that $e \le X(x,x)$ and $X(x,y) \otimes X(y,z) \le X(x,z)$ for all $x,y,z \in X$.

The exercise above shows that from a system of relations, we get an X which 'has identity and composition'. The next step is to find out whether these properties are sufficient to get a system of relations via $R_v = \{(x,y) \mid v \le X(x,y)\}$.

Exercise 2.8. Show that the conditions $e \le X(x,x)$ and $X(x,y) \otimes X(y,z) \le X(x,z)$ are sufficient to guarantee that $R_v = \{(x,y) \mid v \le X(x,y)\}$ is a system of relations.

For future reference, we give a name to structures X with identity and composition as in the previous two exercises. Overloading X in the definition below to a set and a function and a pair consisting of a set and function is convenient and will not create ambiguities as these three different uses of the same symbol are distinguished by their different types.

Definition 2.9 (categories enriched over a quantale)**.** Let $(\Omega, e, \otimes)$ be a commutative unital quantale. Let X be a set and let $X : X \times X \to \Omega$. Then X is called a *category enriched over* Ω if

$$e \le X(x,x) \qquad \text{(identity)}$$
$$X(x,y) \otimes X(y,z) \le X(x,z) \qquad \text{(composition)}$$

for all $x,y,z \in X$.

Exercise 2.10. In case the quantale in question is the two element one, what are the common mathematical terms for enriched category, identity and composition?

We discussed that for every system of relations there is a corresponding enriched category and that every enriched category gives rise to a system of relations. To complete the analysis, we also want to know the following.

Exercise 2.11. Show that starting from an enriched category and going to the corresponding system of relations and back yields the original enriched category.

Finally, to consolidate the findings of this section, read up on metric spaces and summarise what you learned:

Exercise 2.12. Revise the notion of a metric space. Let Ω be the non-negative reals with ∞ as in Example 1.3. Show that every metric space is a category enriched over Ω. Discuss how the categorical notion is more general than a metric space. Speculate about the importance of this generalisation. Read the beginning of Lawvere's [1] and report back something interesting.

3 Examples with quantales of size 3

Let $\mathfrak{3} = \{0 < 1 < 2\}$ be equipped with a monotone, commutative operation satisfying $0 \otimes i = 0$ for all $i \in \mathfrak{3}$. As we know from Exercises 2.8 and 2.11, a category X enriched over $\mathfrak{3}$ can be understood as a system of relations

$$R_2 \subseteq R_1.$$

satisfying the conditions of Definition 2.1. We will see shortly, that there are exactly three ways of extending $\mathfrak{3}$ to a quantale. Two of them have a special property that is the topic of the next exercise.

Exercise 3.1. Show that the relations R_i are transitive if and only if i is *idempotent*, that is, $i \otimes i = i$.

Example 3.2. There are exactly two ways of turning $\mathfrak{3}$ into a commutative quantale with an idempotent tensor, determined by choosing $1 \otimes 2 = 2$ or $1 \otimes 2 = 1$. We write out the tables of tensor and internal hom for future reference.

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$[-,-]$	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

$\otimes$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$[-,-]$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

The reason why we are able to get away with not writing out the entire tables is the subject of the next exercise.

Exercise 3.3. Show that in every quantale, we have: (i) If $e = \top$ then $x \to x = \top$. (ii) If $a \leq e$ then $a \otimes b \leq b$ and $[a, b] \geq b$.

The Ω-enriched categories for the three-element quantales $\Omega = (\underline{3}, \otimes, e, [-,-])$ can be interpreted, following [2, 3], as well-known models of concurrency given by sets equipped with two relations (in the terminology of the $(X_r)_{r\in\Omega}$ above, the relation X_0 is reduntant since 0 is bottom). We will explore this in the following.

Remark 3.4 (Prossets)**.** The first case, which is determined by $e = 1$, accounts for the *prossets* of Gaifman and Pratt [4]. Explicitly, the objects of an Ω-category can be seen as events subject to a schedule, endowed with a preorder $x \leq y$ given by X_1 (with the interpretation that "x happens no later than y") and a binary relation $x \prec y$ given by X_2 (which is intended to mean "x happens strictly earlier than y"). $X_2 \subseteq X_1$ says that strict precedence implies weak precedence, while the multiplication law $1 \otimes 2 = 2$ accounts for the prosset-law that $x \leq y \prec z \leq w$ implies $x \prec w$.

Exercise 3.5. Read enough of Gaifman and Pratt [4] to ask a question or report back something interesting.

Remark 3.6 (Time and Causality)**.** The second case, which is determined by $e = 2$, is due to Gaifman [5]. The relation $x <^t y$ given by X_1 is interpreted as "x precedes y in time" and the relation $x <^c y$ given by X_2 is interpreted as "x causally precedes y". $X_2 \subseteq X_1$ captures that causal precedence implies temporal precedence and the multiplication law $1 \otimes 2 = 1$ reflects that $x <^t y <^c z <^t w$ implies $x <^t w$.

Exercise 3.7. Read enough of Gaifman [5] to ask a question or report back something interesting.

A famous example with a non-idempotent tensor on the three chain is as follows.

Example 3.8. The Łukasiewicz chain $Ł_2 = \{0 < 1/2 < 1\}$ with $e = 1$ and $\max(0, x + y - 1)$ as tensor, now written $x \odot y$, has Ω-enriched categories that can be described as follows. A carrier X with two relations $R_1 \subseteq R_{1/2}$ such that R_1 is reflexive and transitive. [3] We can assign a truth-value

[3] Note that $1/2 \odot 1/2 = 0$ so that $R_{1/2}; R_{1/2} \subseteq R_0$ but since 0 is bottom, we have that $R_0 = X \times X$, which imposes no constraint on composition of pairs of $R_{1/2}$.

to pairs according to $X(x,y) = \bigvee\{r \mid (x,y) \in r\}$. As usual, we put $x \leq_X y \Leftrightarrow X(x,y) \geq 1$. [4] While $v \odot -$ will interpret $v \star -$, the meaning of $v \rhd b$ is given by $\min(1, 1+b-v)$. [5]

$\otimes$	0	1/2	1
0	0	0	0
1/2	0	0	1/2
1	0	1/2	1

$[-,-]$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

Exercise 3.9. Read enough of the wikipedia article on Lukasiewicz logic to ask a question or report back something interesting.

A Some background on category theory

We review some background material. It is not needed to solve the exercises, but may clarify some of the development.

A.1 A Theory of Structures

One question that arises in Section 1 is why a quantale is defined as a complete join semi-lattice (plus some extra structure) if any complete join semilattice also has all meets. Wouldn't it be conceptually clearer to ask that a quantale is a complete lattice (plus the extra structure)?

The answer is no and the reason is that we should define structures with a notion of structure preserving map in mind. And if we talk about quantales, we think of the join-preserving maps, but do not (usually) care about meet-preserving maps. One reason for the latter is that with quantales we are interested in tensor-preserving maps rather than in meet-preserving maps.

This explanation is not satisfactory to a beginning student of quantales, but I don't want to make a larger detour here. Instead, I want to make join-preserving maps part of the definition of a quantale. The cleanest way to do this is in the language of category theory.

A category can be understood as a graph with identities and associative composition or as a many-sorted monoid.

Definition A.1. A category $\mathcal{C}$ has a set of "objects" $A, B, \ldots \in \mathcal{C}$ and for each pair (A, B) of objects a set $\mathcal{C}(A, B)$ of "arrows". For each object A there is a distinguished arrow id_A called the identity of A. For all objects A, B, C there is a function $\mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$. The image of a pair (f, g) of arrows is denoted by $f;g$ or, more commonly, by $g \circ f$, pronounced g after f. Identities and composition satisfy $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B$, $g : B \to C$, $h : C \to D$, where $f : A \to B$ abbreviates $f \in \mathcal{C}(A, B)$.

Example A.2. Set is the category of sets and functions. Quant is the category that has quantales as objects and has as arrows the functions that preserve joins and multiplication.

The notion of category allows us to develop a **structure independent theory of structure**. Instead of settling for a particular notion of structure, one axiomatises structure indirectly by axiomatising properties of structure-preserving maps. So far, we only required that, whatever the notion of structure in question, there are so-called identity maps, which are structure preserving and that structure-preserving are closed under an operation called composition.

A.2 Isomorphisms

Informally, two mathematical objects are isomorphic if they have the same structure. As a first example of category theory as an axiomatic theory of structure, we define the notion of isomorphism.

[4] Intuitively, if x implies y only up to 1/2 and y implies z only up to 1/2 then x implies z only up to 0 (ie not at all).

[5] $a \leq v \rhd b$ iff $v \odot a \leq b$ iff $\max(0, v+a-1) \leq b$ iff $v + a \leq 1 + b$ iff $a \leq 1 + b - v$ iff $a \leq \min(1, 1+b-v)$.

Definition A.3. Let $\mathcal{C}$ be a category. An arrow $f : A \to B$ in $\mathcal{C}$ is an isomorphism if there is an arrow $g : B \to A$, called the inverse of f, such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. We say that A and B are isomorphic, written as $A \cong B$, if there exists an isomorphism $A \to B$.

It is an important idea of mathematics that we should consider isomorphic objects as equal. This means that any mathematical construction we may want to perform on mathematical objects should be invariant under isomorphism. This is closely related to how we use an API in computer programming: The programs we develop should be independent on how the interface we are using is implemented.

The next example illustrates that whether two structures are isomorphic or not depends on the structure under current consideration.

Example A.4. In the category Set, the integers and the rationals are isomorphic. In the category of ordered sets, they are not.

So far we did not use that composition in a category is associative. Of course, in order to be able to consider isomorphic objects as equal, we need the following property.

Exercise A.5. Show that the composition of two isomorphisms is an isomorphism. Conclude that $\cong$ is an equivalence relation on the objects of the category.

The first motivation for this appendix was that it, hopefully, shed some light on the fact that it makes a difference whether one defines quantales via complete join semi-lattices or via complete lattices.

The second motivation is that, in Section 2, we show that there is an bijection between the class of categories enriched over Ω and the class of systems of relations indexed by Ω. In the light of Definition A.3, we now summarise Exercises 2.6 - 2.11 as stating the existence of functions F and G

$$\Omega\text{-cat} \underset{G}{\overset{F}{\rightleftarrows}} \Omega\text{-SoR} \tag{1}$$

such that

- FX is an SoR
- GR is an enriched category
- $F \circ G = \mathrm{id}_{\Omega\text{-SoR}}$
- $G \circ F = \mathrm{id}_{\Omega\text{-cat}}$

If it wouldn't lead us too far afield, the proper mathematical treatment of the situation above would be to define arrows between enriched categories and also between SoR's and then to show that F and G form an isomorphism of categories. For now, hopefully, this clarifies what is at stake in Exercises 2.6 to 2.11.

Exercise A.6. (provisional) Is condition 4 of Definition 2.1 needed to make (1) work? Where exactly is it used?

B Representation Theory of Boolean Algebras

Definition B.1 (Boolean Algebra)**.** A Boolean algebra $(A, 0, 1, \neg, \wedge, \vee)$ is a set A with two constants, a unary operation $\neg$ and two binary operations $\wedge$ and $\vee$, satisfying

...

In general lattices we write $\bot, \top$ for bottom and top, but in Boolean algebras it is more common to write $0, 1$ instead.

In a lattice, b is called a *complement* of a if $a \wedge b = \bot$ and $a \vee b = \top$.

Exercise B.2. Fill in the axioms above. In words, a Boolean algebra is a distributive lattice in which every element has a complement. Show that in a distributive lattice complements are unique. Give an example of a (necessarily non-distributive) lattice in which complements are not unique.

Subsets ordered by inclusion clearly form a lattice. More interesting is that they are distributive:

Exercise B.3 (Powerset, Venn Diagrams)**.** Let $\mathcal{P}X$ be the powerset (=set of subsets) of X ordered by inclusion with union, intersection, and complement. Show that every powerset is a Boolean algebra. More generally, every subset of a powerset closed under unions and complements is a Boolean algebra.

It is convenient to call powerset algebras concrete Boolean algebras and general Boolean algebras abstract Boolean algebras. The previous exercise shows that all powerset algebras are Boolean algebras. In other words, the concrete Boolean algebras form a subclass of the (abstract) Boolean algebras.

The last observation raises the question whether there are laws of logic satisfied by all concrete Boolean algebras but not by all abstract Boolean algebras. In other words, are the axioms of Boolean algebra complete wrt their set-theoretic interpretation? A negative answer would be disturbing, since the Boolean operations $\neg, \wedge, \vee$ are intended to capture complement, intersection and union.

Stone's representation theorem for Boolean algebras guarantees that the abstract notion of Boolean algebra captures exactly powersets with union, intersection and complement. In particular, if a property in the language of $(0, 1, \neg, \wedge, \vee)$ holds for all concrete Boolean algebras (or even just the two element Boolean algebra), then it does hold for all Boolean algebras.

We state the theorem only for finite algebras, but it does hold in general, with the appropriate generalisation of "atoms". But first we need to define what atoms are.

Definition B.4 (Atom)**.** Let A be a Boolean algebra. $a \in A$ is an atom if $a \not\leq b \Rightarrow a \wedge b = 0$ for all $b \in A$.

Exercise B.5. Show that a is an atom iff there is no element strictly in between $\bot$ and a.

Exercise B.6 (Characteristic Functions)**.** Let $2 = \{0, 1\}$. Write 2^X for the set of functions $X \to 2$. Show that the truth tables for $\wedge, \vee, \neg$, if extended 'pointwise' to functions, endow 2^X with the structure of a Boolean algebra. Show that $\mathcal{P}X \cong 2^X$, that is, that they are isomorphic as Boolean algebras.

Theorem B.7 (Representation Theorem)**.** *Every finite Boolean algebra is isomorphic to the powerset of its atoms.*

Exercise B.8. Let A be a Boolean algebra and X its set of atoms. Write 2^X for the Boolean algebra of subsets of X. Define $f : A \to 2^X$ via $f(a) = \{x \in X \mid x \leq a\}$. Show that f is bijective and a homomorphism.

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